Optimal LQG Control Across Packet-Dropping Links

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Abstract

We examine two special cases of the problem of optimal Linear Quadratic Gaussian control of a system whose state is being measured by sensors that communicate with the controller over packet-dropping links. We pose the problem as an information transmission problem. Using a separation principle, we decompose the problem into a standard LQR state-feedback controller design, along with an optimal encoder-decoder design for propagating and using the information across the unreliable link. Our design is optimal among all causal algorithms for any arbitrary packet drop pattern. Further, the solution is appealing from a practical point of view because it can be implemented as a small modification of an existing LQG control design.

Key words: LQG control, Networked control systems, Packet-dropping links, Separation principle

1 Introduction

Recently, much attention has been directed toward systems which are controlled over a communication link (see, e.g., [1,2] and the references therein). In such systems, the control performance can be severely affected by the properties of the communication channel. Communication links introduce many

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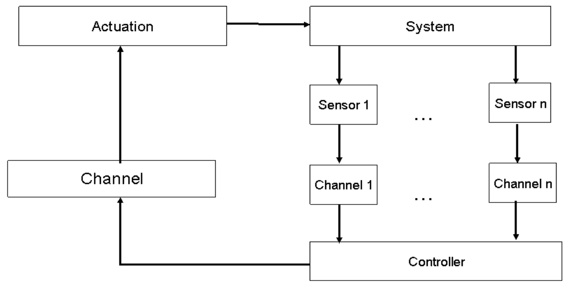

Fig. 1. The architecture of a packet-based control loop. The channels are unreliable and unpredictably drop packets.

potentially detrimental phenomena, such as quantization error, random delays and packet drops to name a few. In extreme cases, poor network performance can even destabilize a nominally stable control loop. Understanding and counteracting these effects will become increasingly important as emerging applications of decentralized control mature.

The above issues have motivated much of the study of networked systems. Beginning with the seminal paper of Delchamps [22], quantization effects have been studied by Tatikonda [23], Elia and Mitter [24], Brockett and Liberzon [26], Hespanha et al. [27], Nair and Evans [25], and many others. The effects of delayed packet delivery have also been considered in many works using various models for the network delay, some representative examples being the works of Nilsson [16], Blair and Sworder [28], Luck and Ray [29] and Zhang et al. [30].

In this note, we are specifically interested in systems communicating over links that randomly drop packets. The nominal system is shown in Figure 1, where the n channels represent communication links or networks that randomly erase packets being communicated from the sensors to the controller. In particular, we discuss two special cases of the problem.

(1) Case C_1: There is only one sensor (and one channel) present.
(2) Case C_2: There are 2 sensors present. However, while channel 1 drops packets randomly, channel 2 transmits all packets.

While the case C_1 is important in its own right, it is also the basic system we need to understand for more general systems with multiple plants, sensors and controllers. Preliminary work in this area studied stability of systems utilizing lossy packet-based communication, as in [31,18,30]. Performance of such systems was analyzed by Seiler in [18] and by Ling and Lemmon in [32]

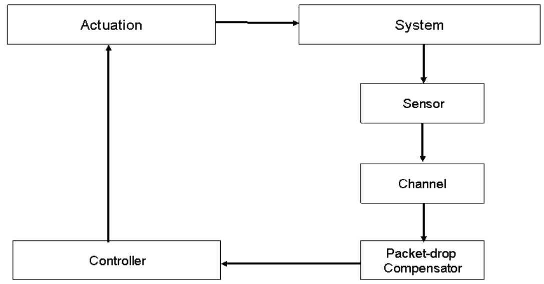

Fig. 2. The usual architecture for compensation for packet drops by the link.

assuming certain statistical dropout models. Approaches to compensate for the data loss have also been proposed. Nilsson [16] proposed two approaches for compensation for data loss in the link by the controller, namely keeping the old control or generating a new control by estimating the lost data, and presented an analysis of the stability and performance of these approaches. Hadjicostis and Touri [20] analyzed the performance when lost data is replaced by zeros. Ling and Lemmon [14,32] posed the problem of optimal compensator design for the case when data loss is independent and identically distributed (i.i.d.) as a nonlinear optimization. Azimi-Sadjadi [3] took an alternative approach and proposed a sub-optimal estimator and regulator to minimize a quadratic cost. Schenato et al. [8] and Imer et al. [13] extended this approach further to obtain optimal controllers when the packet drops were i.i.d. The related problem of optimal estimation across a packet-dropping link was considered by Sinopoli et al in [19] and extended by Gupta et al in [15].

However, most of the designs proposed in these references aim at designing a packet-loss compensator as shown in Figure 2. Most works assume a communication link to be present only between the controller and the actuator. The compensator accepts those packets that the link successfully transmits and comes up with an estimate for the time steps when data is lost. This estimate is then used by the controller. Our work takes a more general approach by seeking the LQG optimal control for this packet-based problem. In particular, for the case C_1, our architecture is as shown in Figure 3. Recognizing that the problem is of making sure that the controller has access to the maximal possible information set (hence an information *transmission* problem), we introduce an encoder at the sensor end. The compensator then effectively becomes a decoder for the information being transmitted over the link. We jointly design the controller, the encoder and the decoder to solve the optimal LQG problem. Even though sensors equipped with wireless or network communication capabilities will likely have some computational power available,

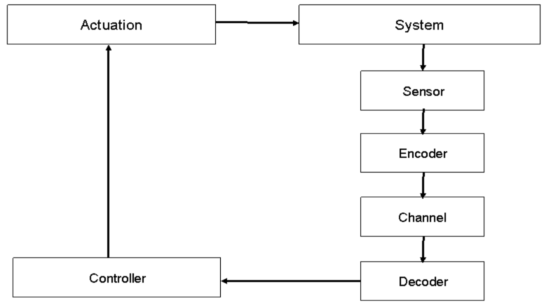

Fig. 3. The structure of our optimal LQG control solution (Case C_1).

we still look for encoding and decoding algorithms that are recursive in structure. Recursive algorithms require a constant amount of memory, processing and transmission and hence will not overwhelm the resources available at the device level.

There does not appear to be existing work dealing with the case C_2 specifically. We encounter this case in our work on the multi-vehicle wireless testbed [10]. In the testbed, each vehicle is equipped with an on-board gyro. In addition, each vehicle also obtains measurements from an overhead camera. While the gyro-controller link is hard-wired and hence does not drop packets, the camera communicates to the controller over a wireless link that randomly drops packets. Thus this situation is identical to the case C_2. Our solution to this problem again adopts the philosophy of using some computation at the sensor end to combat the effects of the channels. Our architecture is as shown in Figure 4. We again provide recursive yet optimal designs of the encoders, the decoder and the controller.

Since the focus of the paper is on presenting the idea of information preprocessing to counter channel effects in networked control, to simplify the presenation, for most of the paper we will assume a channel between the sensor and the controller only. We will, however, revisit the problem for a channel being present between the controller and the actuator in Section 3.2.3. We will see that most of the results presented in the paper can easily be carried over to that case.

The main contribution of the paper is posing and solving the problem of LQG control across a communication channel as an information transmission problem. Because of the real-time constraint of the control problem, asymptotic information theoretic block coding type operations cannot be used. For the specific cases C_1 and C_2, we obtain the optimal encoding and decoding strate-

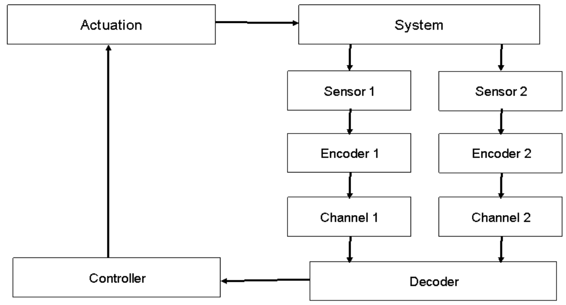

Fig. 4. The structure of our optimal LQG control solution for the two-sensor case (Case C_2).

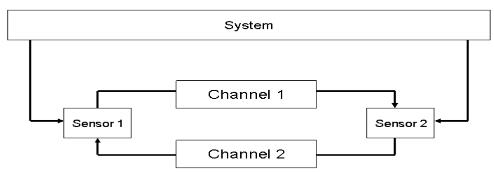

Fig. 5. Structure of the joint estimation problem (Case C_3).

gies for the purpose of LQG control. The strategies are optimal in the sense that no other causal strategy can lead to a better performance even though our strategies only require bounded memory, processing and transmission. As an intermediate step, we also solve the following problem, that we refer to as case C_3.

- Case C_3: Suppose, as shown in Figure 5, two sensors are estimating a process jointly while communicating over links that drop packets stochastically. What information should the sensors exchange?

Related work to this problem has dealt with fusion of data from multiple sensors and track-to-track fusion. A usual starting point for such works is an attempt to decentralize the Kalman filter as, e.g., in [33]. However this approach requires that data about the global estimate be sent from the fusion node to the local sensors. This difficulty was first overcome in [34,35] and further in [11] where both the measurement and time update steps of the Kalman filter were decentralized. Alternative approaches for data fusion from many nodes include using the Federated filter [6], Bayesian methods [9], a scattering framework [21], algorithms based on decomposition of the information form of the Kalman filter [17] and so on.

However these approaches assume a fixed communication topology among the nodes with a link, if present, being perfect. In our case, packets of information are dropped randomly by the communication channels. This random loss of information reintroduces the problem of correlation between the estimation errors of various nodes [4] and renders the approaches proposed in the literature as sub-optimal. An approach to solve this problem was proposed in [5] in the context of track-to-track fusion through exchange of state estimates based on each sensor's own local measurements but the specific scheme that was used was not proven to be optimal. It was subsequently proven in [7] that the technique was based on an assumption that was not met in general. We wish to address this problem of finding the optimal global estimate for each node in the case when there are communication channels present between the nodes and packets of information are being randomly dropped. Once again, we disallow approaches such as transmitting all the measurements taken by each node each time communication is possible because they can potentially entail transmitting arbitrarily large amounts of data. Instead we will propose a recursive yet optimal strategy.

This paper is organized as follows. We begin in the next section by posing the LQG problem in a packet-based setting. We then discuss a separation between control and estimation costs, and present an optimal solution to the estimation problem. We discuss some extensions to the algorithm. Finally, we analyze the stability of our system and compare its performance with some other approaches in the literature.

2 Problem Formulation

Consider a discrete-time linear system evolving according to

$$x_{k+1} = Ax_k + Bu_k + w_k, \tag{1}$$

where $x_k \in \mathbf{R}^n$ is the process state, $u_k \in \mathbf{R}^m$ is the control input and w_k is process noise assumed to be white, Gaussian, and zero mean with covariance matrix Q_w [1]. The initial condition x_0 is assumed to be independent of w_k and to have mean zero and covariance matrix Q_0. The state of the plant is

[1] The results continue to hold for time-varying systems, but we consider the time-invariant case to simplify the presentation.

measured by two sensors according to the equations

$$y_k^i = C^i x_k + v_k^i \qquad i = 1, 2. \tag{2}$$

The measurement noises v_k^i's are assumed white, zero-mean, Gaussian (with covariance matrix Q_v^i) and independent of the plant noise w_k and of each other. Note that substituting $C^2 = 0$ and $Q_v^2 = 0$ would reduce the case C_1 to be a special case of C_2. Hence, from now on, we will carry out the derivation for case C_2 only and adapt the results for the case of one sensor. Each sensor communicates its own measurements (or some function of the measurements) to the controller. We impose the constraint that the function communicated should be a finite vector, whose size does not increase with time. Sensor 1 communicates over channel 1 that randomly drops packets while sensor 2 utilizes channel 2 that is perfect. For the moment we ignore delays and packet reordering in channel 1; it will be shown that these effects can be accounted for with time-stamping and a slight modification to our design. Hence at each time step k,

- A packet containing some function of the measurements is created at both the sensors. We do not specify in advance what data these packets will contain.
- The packets are sent across the link.
- The packet over channel 1 is either received instantaneously, or dropped, probabilistically.

The packet dropping in channel 1 is a random process. We refer to individual (i.e. deterministic) realizations of this random process as *packet drop sequences*. The packet drop sequence P is a binary sequence $\{\eta_k\}_{k=0}^{\infty}$ in which η_k takes the value "*received*" if the link delivers the packet at time step k, and "*dropped*" otherwise.

We assume sufficient bits per packet and a high enough data rate so that quantization error is negligible [2] . We also assume that enough error-correction coding is done within the packets so that the packets are either dropped or received without error. The absolutely optimal LQG performance achievable is obviously given by the classical LQR controller/Kalman estimator pair. However, this design does not respect the packetized nature of the communication. Specifically, the controller requires continual access to the Kalman filter output, which in turn requires continual access to the measurements from both the sensors. This access might not be always possible because of data loss

[2] This assumption merely means that a sufficient number of bits is available so that the effect of quantization error is swamped by the effect of the process and the measurement noises. We do not assume an infinite number of bits, so that strategies based on interleaving of bits to transmit an infinite amount of data are not admissible.

in the communication link. In order to make the class of controllers that are allowed more precise, we introduce the following terminology. Denote by s_k^i the finite vector transmitted from the sensor i to the controller at time step k. By causality, s_k^i can depend (possible in a time-varying manner) on y_0^i, y_1^i, $\cdots$, y_k^i, i.e., $s_k^i = f_k^i\left(y_0^i, y_1^i, \cdots, y_k^i\right)$. The *information set*, I_k available to the controller at time k is the union of two sets I_k^1 and I_k^2 defined by

$$I_k^1 = \{s_j^1 | \forall j \text{ s.t. } \eta_j = \text{ received}\} \qquad\qquad I_k^2 = \{s_j^2 | \forall j = 0 \cdots k\}$$

Also denote by $t_l(k) \leq k$ the last time-step at which a packet was delivered over link L_1. That is

$$t_l(k) = \max\{j \leq k \mid \eta_j = \text{``\textit{received}''}\}.$$

The *maximal information set*, $I_k^{\max}$ at time-step k is then the union of I_k^2 and the set $I_k^{1,\max}$ defined by

$$I_k^{1,\max} = \{y_j^1 \mid 0 \leq j \leq t_l(k)\}.$$

The maximal information set is the largest set of output measurements on which the control at time-step k can depend. In general, the set of output measurements on which the control depends will be less than this set, since earlier packets, and hence measurements, may have been dropped. As stated earlier, the only restriction we impose is that the vectors s_k^i not increase in size as k increases. We will call the set of f_k^i's which fulfill this requirement as $\mathbf{F}$. Without loss of generality, we will only consider information-set feedback controllers, i.e., controllers of the form $u_k = u(I_k, k)$. The control input u_k is transmitted to the actuator and applied to the process. We denote the set of control laws allowed by U. We shall assume perfect knowledge of the system parameters A, B, C, Q_w and Q_v^i's at the controller. Moreover we assume that the controller (and the decoder) have access to the previous control signals u_0, u_1, $\cdots$, u_{k-1}.

We can thus pose the packetized LQG problem as:

$$\min_{u \in U, f^i \in \mathbf{F}} J_K(u, f^i, P_1, P_2) = E\left[\sum_{k=0}^{K}\left(u_k^T Q^c u_k + x_k^T R^c x_k\right) + x_{K+1}^T P_{K+1}^c x_{K+1}\right]. \tag{3}$$

Here K is the horizon on which the plant is operated and the expectation is taken over the uncorrelated variables x_0, $\{w_k\}$ and $\{v_k^i\}$. Note that the cost functional J above depends on the random packet-drop sequence P. However, we do *not* average across packet-drop processes; *the solution we will present is optimal for arbitrary realizations of the packet dropping process.* We now present our solution to the problem.

3 Optimal Encoder and Decoder Design

Recall that we wish to construct the optimal control input based on the information set $I_k^{\max}$, but we have not yet specified how to design f_k^i's that will allow the controller to compute that. If channel 1 does not drop packets, sending the current measurement y_k^i in the current packets is sufficient. When channel 1 randomly drops packets, a naíve solution would be to send the entire history of the output variables at each time step. This would certainly be an optimal solution, however, as mentioned earlier, this is not allowed since it requires increasing data transmission as time evolves. Surprisingly, we can achieve performance equivalent to the naíve solution using a constant amount of transmission, and memory. To this end, we first state the following separation principle.

Proposition 1 (Separation) *Consider the packet-based optimal control problem defined in section 2. Suppose that both the sensors transmit all the previous measurements at every time step, so that the decoder has access to the maximal information set $I_k^{\max}$ at every time step k. Then, for an optimizing choice of the control, the control and estimation costs decouple. Specifically, the optimal control input at time k is calculated by using the relation*

$$u_k = \hat{\bar{u}}_{k|I_k^{\max}} = -\left(R_{e,k}^c\right)^{-1} B^T P_{k+1}^c A \hat{x}_{k|I_k^{\max}},$$

where $\bar{u}_k$ is the optimal LQ control law while $\hat{\alpha}_{k|I_k^{\max}}$ denotes the lms estimate of α given the information set $I_k^{\max}$ and the previous control laws $u_0, \cdots, u_{k-1}$.

PROOF. The proof is along the lines of the standard separation principle (see, e.g., [12]) and is omitted for space constraints. □

There are two reasons this principle is useful to us:

(1) The controller design part of the problem is now solved. The optimal controller is the solution to the LQ control problem.
(2) The optimal controller does not need to have access to the information set $I_k^{\max}$ at every time step k. The encoders and the decoder only need to ensure that the controller receives the quantity $\hat{\bar{u}}_{k|I_k^{\max}}$, or equivalently, $\hat{x}_{k|I_k^{\max}}$.

We now propose an algorithm that requires a constant amount of memory and transmission, yet allows the controller to have access to $\hat{x}_{k|I_k^{\max}}$ at every time step.

Let $\hat{x}^i_{k|l}$ denote the estimate of x_k based on all the measurements of sensor i up to time l and all previous control inputs. Denote the corresponding error covariance by $P^i_{k|l}$. Also denote by $\bar{x}^i_{k|l}$ the estimate of x_k based on all the measurements of sensor i up to time l while assuming that no control input was applied as x_k evolved according to (1). $\bar{x}^i_{k|l}$ can be evaluated through a filter that is identical to a Kalman filter except for the application of the control input during the time update step. We will call such a filter a modified Kalman filter. Note that the calculation of the quantity $P^i_{k|l}$ is identical for both the Kalman filter and the modified Kalman filter even though $P^i_{k|l}$ does not stand for the estimate error covariance in the case of the modified Kalman filter.

(1) *Encoder for sensor 1:* At each time step k,
- Obtain measurement y^1_k and run a local modified Kalman filter to obtain $\bar{x}^1_{k|k}$ and $P^1_{k|k}$.
- Calculate $\lambda^1_k = \left(P^1_{k|k}\right)^{-1} \bar{x}^1_{k|k} - \left(P^1_{k|k-1}\right)^{-1} \bar{x}^1_{k|k-1}$.
- Calculate global error covariance matrices $P_{k|k}$ and $P_{k|k-1}$ using

$$\begin{aligned} \left(P_{k|k}\right)^{-1} &= \left(P_{k|k-1}\right)^{-1} + \left(C^1\right)^T \left(Q^1_v\right)^{-1} \left(C^1\right) + \left(C^2\right)^T \left(Q^2_v\right)^{-1} \left(C^2\right) \\ P_{k|k-1} &= AP_{k-1|k-1}A^T + Q_w. \end{aligned}$$

- Obtain $\gamma_k = \left(P_{k|k-1}\right)^{-1} A_{k-1}P_{k-1|k-1}$.
- Finally calculate $i^1_k = \lambda^1_k + \gamma_k i^1_{k-1}$ with $i^1_{-1} = 0$ and transmit it.

(2) *Encoder for sensor 2:* At each time step k, transmit the measurement y^2_k.

(3) *Decoder:* At each time step k,
- Use y^2_k to come up with i^2_k using an algorithm similar to the one followed by the encoder for sensor 1.
- Maintain a local variable $\hat{x}^{dec}_k$ which is updated as follows.
  (a) If $\eta_k = \textit{received}$, both links L_1 and L_2 have successfully transmitted packets. In that case, calculate $\psi_k = \left(P_{k|k-1}\right)^{-1} Bu_{k-1} + \gamma_k \psi_{k-1}$ with $\psi_0 = 0$ and obtain the estimate through

$$\left(P_{k|k}\right)^{-1} \hat{x}^{dec}_k = i^1_k + i^2_k + \psi_k.$$

  (b) If $\eta_k = \textit{dropped}$, only L_2 has transmitted the packet. In this case, propagate the estimate $\hat{x}^{dec}_{k-1}$ using the measurement y^2_k and the control u_{k-1} through a Kalman filter.

  The variable $\hat{x}^{dec}_k$ is the estimate of the decoder.

Proposition 2 (Optimal Estimation) *In the above algorithm,* $\hat{x}^{dec}_k = \hat{x}_{k|I^{max}_k}$.

PROOF. Consider a centralized filter that has access to measurements from a sensor of the form

$$y_k = Cx_k + v_k$$

where

$$C = \begin{bmatrix} C^1 \\ C^2 \end{bmatrix} \qquad v_k = \begin{bmatrix} v_k^1 \\ v_k^2 \end{bmatrix}. \tag{4}$$

Let R be the covariance matrix of the noise v_k. Since R is block-diagonal, the measurement update equations of the Kalman filter are

$$\begin{aligned}
\left(P_{k|k}\right)^{-1} &= \left(P_{k|k-1}\right)^{-1} + C^T R^{-1} C \\
&= \left(P_{k|k-1}\right)^{-1} + \sum_i \left[\left(P^i_{k|k}\right)^{-1} - \left(P^i_{k|k-1}\right)^{-1}\right] \\
\left(P_{k|k}\right)^{-1} \hat{x}_{k|k} &= \left(P_{k|k-1}\right)^{-1} \hat{x}_{k|k-1} + C^T R^{-1} y_k \\
&= \left(P_{k|k-1}\right)^{-1} \hat{x}_{k|k-1} + \sum_i \left[\left(P^i_{k|k}\right)^{-1} \hat{x}^i_{k|k} - \left(P^i_{k|k-1}\right)^{-1} \hat{x}^i_{k|k-1}\right].
\end{aligned}$$

Recognizing that the time update equations are

$$P_{k|k-1} = AP_{k-1|k-1}A^T + Q_w, \qquad \hat{x}_{k|k-1} = A\hat{x}_{k-1|k-1} + Bu_{k-1},$$

we can write

$$\left(P_{k|k}\right)^{-1} \hat{x}_{k|k} = \sum_i I^i_k + \Psi_k.$$

The term I^i_k is the contribution of the measurements of the i-th sensor and is given by

$$I^i_k = \Lambda^i_k + \Gamma_k \Lambda^i_{k-1} + \Gamma_k \Gamma_{k-1} \Lambda^i_{k-2} + \cdots + \left(\Gamma_k \Gamma_{k-1} \cdots \Gamma_1\right) \Lambda^i_0,$$

where

$$\Lambda^i_k = \left(P^i_{k|k}\right)^{-1} \bar{x}^i_{k|k} - \left(P^i_{k|k-1}\right)^{-1} \bar{x}^i_{k|k-1}, \qquad \Gamma_k = \left(P_{k|k-1}\right)^{-1} AP_{k-1|k-1}.$$

The term Ψ_k is the contribution of the control input and can be calculated recursively through

$$\Psi_k = \left(P_{k|k-1}\right)^{-1} Bu_{k-1} + \Gamma_k \Psi_{k-1},$$

with $\Psi_0 = 0$. In the above derivation, we have used the fact that x_0 was zero mean and thus $\hat{x}_{0|-1} = 0$. The covariance matrices can be calculated offline. Thus, the information needed from sensor i at time step k is precisely I^i_k. Now for the case when $\eta_k = \textit{received}$, the decoder in the algorithm has access to i^1_k and i^2_k that are the same as I^1_k and I^2_k. Thus it can calculate the centralized Kalman filter output $\hat{x}_{k|k}$ which is $\hat{x}_{k|I_{max}}$. For the case when $\eta_k = \textit{dropped}$, the decoder propagates the best Kalman filter estimate $\hat{x}_{k-1|k-1}$ with sensor 2's measurement. Thus in this case too, $\hat{x}_k^{dec} = \hat{x}_{k|I^{max}}$ □

Proposition 2 presents the solution to the estimation problem in the case C_3 mentioned in Section 1 since we can use an encoder and a decoder described in the algorithm at each sensor. Moreover, taken together, Propositions 1 and 2 solve the packet-based LQG control problem posed in Section 2.

Proposition 3 (Optimal Packet-Based LQG Control) *For the packet-based optimal control problem stated in section 2, an LQR state feedback design together with the optimal transmission-estimation algorithm described above achieves the minimum of $J(u, f^i, P)$ for any P.*

Remarks:

(1) Note that the computation and memory required for calculating I_k^i does not grow with time since we can use the recursion $I_k^i = \Lambda_k^i + \Gamma_k I_{k-1}^i$.
(2) The information vector I_k^i 'washes away' the effect of any previous packet losses. If $\eta_k = received$, $\hat{x}_{k|k}$ is calculated as if all the previous measurements from both sensors were available.
(3) We have made no assumption about the packet dropping behavior. The algorithm provides the optimal estimate based on I_k^{max} for an arbitrary packet drop sequence, irrespective of whether the packet drop can be modeled as an i.i.d. process (or a more sophisticated model like a Markov chain) or whether its statistics are known or unknown to the plant and the controller.
(4) We do not assume knowledge of the cost matrices Q^c and R^c at the sensor end. Thus the cost function (and hence the optimal controller) can be changed at will without affecting the sensor/encoder operation. This is important, e.g., in our MVWT work where the matrices Q^c and R^c are user-specified while the encoder code is much harder to change.

3.2 Extensions and Special Cases

We now discuss the application of the algorithm to some special cases and generalizations.

3.2.1 The single sensor case:

For case C_1, the algorithm reduces to the following:

- The encoder (at the sensor end) receives as input the measurement y_k. It runs the modified Kalman filter and transmits the output $\bar{x}_{k|k}$ of this filter across the link.
- The decoder (at the controller end) maintains two variables: a variable ψ_k that takes into account the effect of the control inputs, and a local variable $\hat{x}_k^{dec}$ that is updated as follows:
  · If $\eta_k =$ *received*, the decoder receives i_k, and sets $\hat{x}_k^{dec} = \bar{x}_{k|k} + \psi_k$.
  · If $\eta_k =$ *dropped*, then the decoder implements the linear predictor:

$$\hat{x}_k^{dec} = A\hat{x}_{k-1}^{dec} + \psi_k. \tag{5}$$

3.2.2 Presence of delays:

The solution can readily be extended to the case when the channel applies a random delay to the packet so that packets might arrive at the decoder delayed or even out-of-order, if we assume that there is a provision for time-stamping the packets sent by the encoder. For ease of notation, we present the solution for optimal asynchronous estimation for the case C_1. The case C_2 is similar. At each time step, the decoder will face one of four possibilities, and will update its estimate as described below:

- It receives $\bar{x}_{k|k}$. It uses this to calculate the estimate according to $\hat{x}_k^{dec} = \bar{x}_{k|k} + \psi_k$.
- It does not receive anything. It uses the predictor equation (5) on $\hat{x}_{k-1}^{dec}$.
- It receives $\bar{x}_{m|m}$ while at a previous time step, it has already received $\bar{x}_{n|n}$, where $n > m$. It discards $\bar{x}_{m|m}$ and uses (5) on $\hat{x}_{k-1}^{dec}$.
- It receives $\bar{x}_{m|m}$ and at no previous time step has it received $\bar{x}_{n|n}$, where $n > m$. It uses $\bar{x}_{m|m}$ to calculate $\hat{x}_m^{dec}$ and obtains $\hat{x}_k^{dec}$ through (5).

3.2.3 Channel between the controller and the actuator:

As pointed out in [8,13] if we have a channel between the controller and the plant, the separation principle would still hold, provided there is a provision for acknowledgment from the receiver to the transmitter for any packet successfully received over the channels. Since the decoder is assumed to have access to the control input applied at every time step, it is apparent that our algorithm can easily be generalized to this case. We can also ask the question of the optimal encoder-decoder design of the controller-actuator channel. The optimal decoding at the actuator end will depend on the information that is assumed to be known to the actuator (e.g. the cost matrices Q and R and the measurements from the sensor). Design of the decoder for various information sets is an interesting problem, however it is beyond the scope of this paper.

3.2.4 Multiple packet dropping channels:

The algorithm, as proposed, does not extend to multiple packet dropping channels. The crucial assumption used in the algorithm that prevents this extension is that the encoder for sensor 1 uses the fact that sensor 2 will transmit its information at every time step. For multiple channels, this assumption will not be satisfied. Extension of the algorithm to such cases remains an open problem.

4 Analysis of the Proposed Algorithm

In this section, we model the channel erasures as occurring according to a Markov chain and analyze the stability and performance of our design. Thus the channel exists in either of two states, state 1 corresponding to a packet drop and state 2 corresponding to no packet drop and it transitions probabilistically between these states according to the transition probability matrix Q. Note that i.i.d. drops can be handled by a special choice of Q. We assume strict causality in the Kalman filter used by the encoder. Thus to calculate the estimate of x_k, only the measurements till time step $k-1$ are used. The analysis for the causal case is similar. Finally we assume that (A, B) is stabilizable and the pair (A, C) is detectable, where C is defined in (4). We will denote the Kronecker product of matrices A and B by $A \otimes B$.

We begin with the stability analysis. Denote by y_k the vector formed by stacking y_k^1 and y_k^2. We have three dynamical systems. The plant state x_k evolves as in (1). The state $\hat{x}_k$ of a centralized Kalman filter with access to measurements from both sensors at every time step would evolve as

$$\hat{x}_{k+1} = A\hat{x}_k + Bu_k + K_k^c \left(y_k - C\hat{x}_k\right).$$

Finally the state $\hat{x}_k^{dec}$ of the estimator at the decoder evolves according to

$$\hat{x}_{k+1}^{dec} = \begin{cases} A\hat{x}_k^{dec} + Bu_k + K_k^d \left(y_k^2 - C^2\hat{x}_k^{dec}\right) & \text{channel in state 1} \\ \hat{x}_{k+1} & \text{otherwise.} \end{cases}$$

Denote $e_k = x_k - \hat{x}_k$ and $t_k = \hat{x}_k - \hat{x}_k^{dec}$. Since $u_k = F_k\hat{x}_k^{dec}$, (1) implies

$$x_{k+1} = \left(A + BF_k\right)x_k + w_k - BF_k\left(t_k + e_k\right).$$

Since (A, B) is stabilizable and F_k is the optimum control law, the system would be stable in the bounded covariance sense as long as the disturbances w_k, t_k and e_k have bounded covariances. We assume the noise w_k has bounded covariance matrix. Also e_k has bounded covariance matrices by our detectability assumption. Finally t_k evolves according to

$$t_{k+1} = \begin{cases} \left(A - K_k^d C^2\right) t_k + L^1(e_k) + L^2(v_k^1) + L^3(v_k^3) & \text{channel in state 1} \\ 0 & \text{otherwise,} \end{cases} \tag{6}$$

where $L^n(\beta)$ denotes a term linear in β. Again note that v_k^i's and e_k have bounded covariance. For t_k to be of bounded variance, the Markov jump system of (6) needs to be stable. Finally, since our controller and encoder/decoder design is optimal, if the closed loop is unstable with our design, it is not stabilizable by any other design. We can thus say the following.

Proposition 4 (Stability Condition) *Consider the control problem defined in Section 2 in which the packet erasure channel is modeled as a Markov chain with transition probability matrix $Q = [q_{ij}]$. Let the matrix pair (A, B) be stabilizable and the matrix pair (A, C) be detectable. The system is stabilizable, in the sense that the variance of the state is bounded, if and only if $q_{22}|\lambda_{\max}\left(\bar{A}\right)|^2 < 1$, where $\lambda_{\max}\left(\bar{A}\right)$ is the maximum magnitude eigenvalue of the unobservable part of matrix A when (A, C^2) is put in the observer canonical form. Further, if the system is stabilizable, one controller and encoder/decoder design that stabilizes the system is given in Proposition 3.*

Using the results of [16], we can also calculate the total quadratic cost incurred by the system for the infinite-horizon case (the case when $K \to \infty$ in (3)) if we make the additional assumption that the Markov chain is stationary and regular. We state the result for the case C_1. We consider the cost

$$J_\infty = \lim_{K\to\infty} E\left[x_K^T R^c x_K + u_K^T Q^c u_K\right] = \operatorname{trace}\left(P_x^\infty R^c\right) + \operatorname{trace}\left(P_u^\infty Q^c\right), \tag{7}$$

where $P_x^\infty = \lim_{K\to\infty} E\left[x_K x_K^T\right]$ and $P_u^\infty = \lim_{K\to\infty} E\left[u_K u_K^T\right]$. We see that

$$P_x^\infty = \begin{bmatrix} I & 0 & 0 \end{bmatrix} P^\infty \begin{bmatrix} I \\ 0 \\ 0 \end{bmatrix} \qquad P_u^\infty = F \begin{bmatrix} I & -I & -I \end{bmatrix} P^\infty \begin{bmatrix} I \\ -I \\ -I \end{bmatrix} F^T,$$

where $P^\infty = \tilde{P}_1 + \tilde{P}_2$ and $\tilde{P} = \begin{bmatrix} \operatorname{vec}(\tilde{P}_1)^T & \operatorname{vec}(\tilde{P}_2)^T \end{bmatrix}^T$. Then, it can be shown that $\tilde{P}$ is the unique solution to the linear equation

$$\tilde{P} = \left(Q^T \otimes I\right) \begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix} \tilde{P} + \left(Q^T \otimes I\right) \left(\begin{bmatrix} \pi_1 & 0 \\ 0 & \pi_2 \end{bmatrix} \otimes I \right) G.$$

In the above equation, $A_i = \mathbf{A}_i \otimes \mathbf{A}_i$, and $G = \begin{bmatrix} \mathrm{vec}(G_1)^T & \mathrm{vec}(G_2)^T \end{bmatrix}^T$, where

$$\mathbf{A}_1 = \begin{bmatrix} A+BF & -BF & -BF \\ A-KC & 0 & 0 \\ 0 & -KC & A \end{bmatrix} \qquad \mathbf{A}_2 = \begin{bmatrix} A+BF & -BF & -BF \\ A-KC & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

$$\mathbf{B}_1 = \begin{bmatrix} I & 0 \\ I & -K \\ 0 & -K \end{bmatrix} \qquad \mathbf{B}_2 = \begin{bmatrix} I & 0 \\ I & -K \\ 0 & 0 \end{bmatrix} \qquad G_i = \mathbf{B}_i \begin{bmatrix} Q_w & 0 \\ 0 & Q_v \end{bmatrix} \mathbf{B}_i^T.$$

Example

We now consider some examples to illustrate the performance of our algorithm. First, we consider the example system considered by Ling and Lemmon in [14]. The system evolves as

$$x_{k+1} = \begin{bmatrix} 0 & -2 \\ 1 & -1 \end{bmatrix} x_k + \begin{bmatrix} 2 \\ 1 \end{bmatrix} u_k + \begin{bmatrix} 2 \\ 1 \end{bmatrix} w_k.$$

There is only one sensor of the form

$$y_k = \begin{bmatrix} 0 & 1 \end{bmatrix} x_k.$$

The process noise w_k is zero mean with unit variance and the packet drop process is i.i.d. The cost considered is the steady state output error $\lim_{K\to\infty} y_K^2$. [14] assumes unity feedback when packets are delivered and gives an optimal compensator design when packets are being lost.

On analyzing the system with our algorithm, we observe that our algorithm allows the system to be stable up to a packet drop probability of 0.5 while the optimal compensator in [14] is stable only if the probability is less than 0.25. Also if we analyze the performance we obtain the plot given in Figure 6. The performance is much better throughout the range of operation for our algorithm, even if we assume unity feedback in our algorithm. This shows that the difference in performance is mainly due to the novel encoding-decoding algorithm proposed.

In the next example, we consider the same system being observed through two

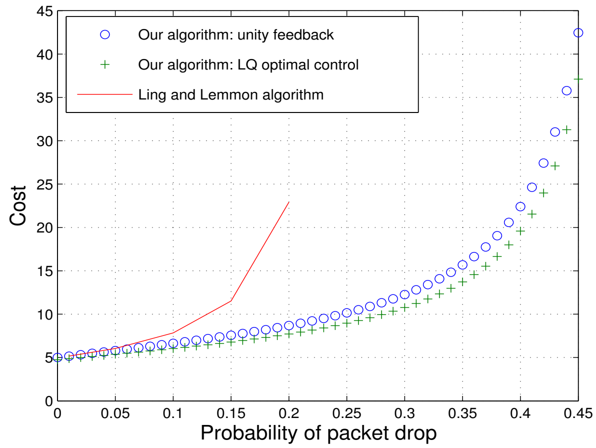

Fig. 6. Comparison of performance for our algorithm with the one obtained using optimal compensator.

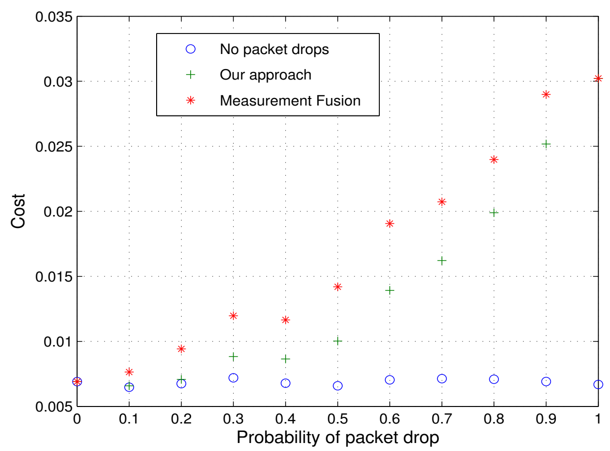

Fig. 7. Comparison of performance for the two sensor case.

sensors of the form

$$y_k^1 = \begin{bmatrix} 1 & 0 \end{bmatrix} x_k + v_k^1 \qquad\qquad y_k^2 = \begin{bmatrix} 0 & 1 \end{bmatrix} x_k + v_k^2.$$

The sensor noises are zero mean with variance 10 and 1 respectively. We consider the cost function $\lim_{K\to\infty} \left(y_K^2\right)^2$. Figure 7 shows the simulated performance of our algorithm as a function of the packet loss probability. We also plot the performance for a hypothetical sensor that received information from both sensors without any packet drop and for a scheme in which sensors exchange only measurements. It can be seen that even in this very simple case, our algorithm can lead to a performance gain of up to 40% over the strategy of using no encoder.

5 Conclusions and Future Work

In this paper, we considered the problem of optimal LQG control when the sensor and controller are communicating across a channel or a network. We modeled the link as a switch that drops packets randomly and proved that a separation exists between the optimal estimate and the optimal control law. For the optimal estimate, we identified the information that the sensor should provide to the controller. This can be viewed as constructing an encoder for the channel. We also designed the decoder that uses the information it receives across the link to construct an estimate of the state of the plant. The proposed algorithm is recursive yet optimal irrespective of the packet drop pattern. For the case of packet drops occurring according to a Markov chain, we carried out stability and performance analysis of our algorithm.

The work can potentially be extended in many ways. One obvious extension is to consider multiple sensors and communication links. Another intriguing possibility is considering the effect of allowing only a limited number of bits in the packet. The work of Sahai [36] seems relevant in this direction. However, from the view of optimal control, this issue has to be examined in greater detail. Extensions to decentralized control are another exciting avenue of research.

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