

# High-Rate Space-Time Codes Motivated by $SU(3)$

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**Abstract**—Fully-diverse constellations, i.e., a set of unitary matrices whose pairwise differences are nonsingular, are useful in multi-antenna communications especially in multi-antenna differential modulation, since they have good pairwise error properties. Recently, group theoretic ideas, especially *fixed-point-free (fpf)* groups, have been used to design fully-diverse constellations of unitary matrices. Here we give systematic methods to design space-time codes which are appropriate for three-transmit-antenna differential modulation. The structures of the codes are motivated by the Lie group  $SU(3)$ . One of the codes, called the AB code, has a fast decoding algorithm using the complex sphere decoder. The diversity products of the codes can be easily calculated and simulated performances show that the codes are better than the group-based codes [1] especially at high rates and as good as the elaborately-designed non-group codes [1].

## 1. INTRODUCTION

It is well known in theory that multiple antennas can greatly increase the data rate and the reliability of a wireless communication link in a fading environment. In practice, however, one needs to devise effective space-time transmission schemes. This is particularly challenging when the propagation environment is unknown to the sender and the receiver, which is often the case for mobile applications when the channel changes rapidly.

A differential transmission scheme called *differential unitary space-time modulation* was proposed in [2], [3], [4], which is well-tailored for unknown continuously varying Rayleigh flat-fading channels. The signals transmitted are unitary matrices. In this scheme the probability of error of mistaking one signal  $S_i$  for another  $S_{i'}$ , at high SNR, is proved to be inversely proportional to  $|\det(S_i - S_{i'})|$ . Therefore the quality of the code is measured by its *diversity product*

$$\xi_C = \frac{1}{2} \min_{S_i \neq S_{i'} \in C} |\det(S_i - S_{i'})|^{\frac{1}{M}} \quad (1)$$

where  $M$  is the number of transmit antennas and  $C$  is the set of all possible signals. The design problem is thus the following: "Given the number of transmit antennas,  $M$ , and the transmission rate,  $R$ , find a set  $C$  of  $L = 2^{MR}$   $M \times M$  unitary matrices, such that the minimum of the absolute value of the determinant of their pairwise differences is as large as possible."

The space-time code designs for three-transmit-antenna system are rare. Till now, some group-based codes and non-group

codes are proposed in [1]. The group-based codes, mainly the  $G_{mr}$  codes and diagonal codes, are rare and do not have good performances for high rates. The design of the non-group codes are very difficult and the decoding of both codes needs exhaustive search. In this paper, we proposed design methods for three-transmit-antenna systems. The codes are motivated by the Lie group  $SU(3)$ . The reasons of analyzing the Lie group  $SU(3)$  are as follows.

The design problem, as just stated, appears to be intractable since first both the signal set and the cost function are non-convex and second, the size of the problem can be huge, especially at high data rates. Therefore, in [1], [5], [6], it was proposed to enforce a group structure on the constellation. This has the advantages of simplifying the diversity product and easy decoding [1], [6]. In [1], all finite fully-diverse constellations that form a group are classified. And also, in [6], it is proved that the only fpf infinite Lie groups are  $U(1)$ , the group of unit-modulus scalars, and  $SU(2)$ , the group of unit-determinant  $2 \times 2$  unitary matrices. However, no good constellations are obtained for very high rates from the finite fpf groups, and constellations based on  $U(1)$  and  $SU(2)$  are constrained to one and two-transmit-antenna systems. As mentioned in [7], to get high rate constellations which work for systems with more than 2 transmit antennas, we relax the fpf condition by considering Lie groups of rank 2. (The rank of a Lie group equals the maximum number of commuting basis elements of its Lie algebra and it can be shown that fpf groups have rank 1.) There are three of them: the Lie group of unit-determinant  $3 \times 3$  unitary matrices  $SU(3)$ , the Lie group of  $4 \times 4$  unitary, symplectic matrices  $Sp(2)$ , and one exceptional Lie group  $G_2$ . Constellations based on  $Sp(2)$ , which can be regarded as an extension of the Alamouti's scheme [8], are designed in [7] and simulation results show that they have good performance. In this work, we analyze  $SU(3)$ , which gives us  $3 \times 3$  constellations.

In this paper, we first explore the structure of matrices in  $SU(3)$  and give a parametrization method for them. Based on the parametrization, we propose two  $3 \times 3$  differential unitary space-time constellations. Simple formulas for the two codes are derived from the diversity products can be calculated in a fast way. Necessary conditions for the full-diversity of the codes are also proved. Our conjecture is that they are also sufficient conditions. Simulation results show that the codes have better performances than the group-based codes

[1] especially at high rates and are as good as the elaborately-designed non-group codes [1]. Another exceptional feature of the code is that the AB code has a fast decoding algorithm based on the complex sphere decoder [9].

#### A. Differential Unitary Space-time Modulation

Consider a wireless communication system with  $M$  transmit antennas and  $N$  receive antennas. The channel is used in blocks of  $M$  transmissions (for more on this model, see [10], [11]), the system equations of block  $\tau$  can be written as:

$$X_\tau = \sqrt{\rho} S_\tau H_\tau + V_\tau$$

Here,  $S_\tau$  denotes the  $M \times M$  transmitted signal whose  $(t, m)$  entry indicates the signal sent by the  $m$ th transmit antenna at time  $t$ .  $H_\tau$  is the  $M \times N$  complex-valued propagation matrix, which is unknown to both the transmitter and the receiver, and its  $(m, n)$  entry is the propagation coefficient between the  $m$ th transmit antenna and the  $n$ th receive antenna and has an iid  $\mathcal{CN}(0, 1)$  distribution.  $V_\tau$  is the  $M \times N$  noise matrix whose  $(t, n)$  entry is the noise at the  $n$ th receive antenna at time  $t$  and has iid  $\mathcal{CN}(0, 1)$  distribution.  $X_\tau$  is the  $M \times N$  received signal matrix. The transmitted power constraint is  $\sum_{m=1}^M \mathbb{E} |s_{tm}|^2 = 1, t = 1, \dots, M$  so  $\rho$  represents the expected SNR at each receive antenna.

In differential modulation, the transmitted matrix  $S_\tau$  at block  $\tau$  equals the product of a unitary data matrix  $V_{z_\tau}$  taken from our signal set  $\mathcal{C}$  and the previously transmitted matrix. In other words,  $S_\tau = V_{z_\tau} S_{\tau-1}$  where  $S_0 = I_M$ . The transmission rate is  $R = \frac{1}{M} \log_2 L$ , where  $L$  indicates the cardinality of our code. Further assume that the propagation environment keeps approximately constant for  $2M$  consecutive channel uses, that is,  $H_\tau \approx H_{\tau-1}$ , we may get the fundamental differential receiver equations [12]

$$X_\tau = V_{z_\tau} X_{\tau-1} + W'_\tau \quad (2)$$

where  $W'_\tau = W_\tau - V_{z_\tau} W_{\tau-1}$ . We can see that the channel matrix  $H$  does not appear in (2). This implies that differential transmission permits decoding without knowing the channel information. The ML decoder of  $z_\tau$  is given by

$$\hat{z}_\tau = \arg \max_{l=0, \dots, L-1} \|X_\tau - V_l X_{\tau-1}\| \quad (3)$$

It is shown in [2] that, at high SNR, the pairwise probability of error (of transmitting  $V_l$  and erroneously decoding  $V_{l'}$ ) has an upper bound that is inversely proportional to the diversity product of the code.

#### II. THE SPECIAL UNITARY GROUP $SU(3)$

**Definition 1:** [13]  $SU(n)$  is the group of complex  $n \times n$  matrices obeying  $U^* U = U U^* = I_n$  and  $\det U = 1$ .

From the definition,  $SU(n)$  is the group of complex  $n \times n$  unitary matrices with determinant 1. It is also known that  $SU(n)$  is a compact, simply-connected Lie group of dimension

$n^2 - 1$  and rank  $n - 1$ . Since we are most interested in the case of rank 2, here we focus on  $SU(3)$ , which has dimension 8. The following theorem on the parametrization of  $SU(3)$  is proved.

**Theorem 1 (Parametrization of  $SU(3)$ ):** Any matrix  $U$  belongs to  $SU(3)$  if and only if it can be written as

$$\begin{bmatrix} 1 & 0 \\ 0 & \Phi \end{bmatrix} \begin{bmatrix} \sqrt{1-|\alpha|^2} & 0 & \bar{\alpha} \\ 0 & 1 & 0 \\ -\alpha & 0 & \sqrt{1-|\alpha|^2} \end{bmatrix} \begin{bmatrix} \Psi & 0 \\ 0 & 1 \end{bmatrix} \quad (4)$$

where  $\Phi, \Psi \in SU(2)$ .  $\alpha$  is a complex scalar with  $|\alpha| \leq 1$ .

$\alpha$  is actually the determinant of the sub-matrix of  $U$  by deleting the first row and the first column. This theorem indicates that any matrix in  $SU(3)$  can be written as a product of three  $3 \times 3$  unitary matrices which are basically  $SU(2)$ . (They are actually  $3 \times 3$  unitary representations of  $SU(2)$  by adding an identity block.) Now let's look at the number of degrees of freedom in  $U$ . Since  $\Phi, \Psi \in SU(2)$ , there are 6 degrees of freedom in them. Together with the 2 degrees of freedom in  $\alpha$ , the dimension of  $U$  is 8, which is exactly the same as that of  $SU(3)$ . Based on (4), we can parameterize matrices in  $SU(3)$  by the entries of  $\Phi, \Psi$  and  $\alpha$ , that is, any matrix in  $SU(3)$  can be identified with a 3-tuple  $(\Phi, \Psi, \alpha)$ . There is also an interesting symmetry in (4): the  $i$ th matrix has an identity block at the  $(i, i)$  entry.

#### III. SPACE-TIME CODES DESIGNS MOTIVATED BY $SU(3)$

From (4) we can see that for any  $U_1(\Phi_1, \Psi_1, \alpha_1)$  and  $U_2(\Phi_2, \Psi_2, \alpha_2)$  in  $SU(3)$ , if any two elements of the 3-tuples are identical, the difference matrix is singular. This is because that each of the matrices in (4) has an identity block which results in that the difference of any two matrices of the kind has a zero block. That is, the identity entries spoil the full diversity of the sets. Therefore, we need to replace the identity entries. Note that the  $U$  in (4) can also be written as the following product,

$$\begin{bmatrix} e^{j\theta} & 0 & 0 \\ 0 & \phi_{11} & \phi_{12}e^{-j\theta} \\ 0 & -\phi_{12} & \phi_{11}e^{-j\theta} \end{bmatrix} \begin{bmatrix} ye^{-j(\theta+\xi)} & 0 & \bar{\alpha}e^{-j(\theta-\xi)} \\ 0 & 1 & 0 \\ -\alpha e^{j(\theta-\xi)} & 0 & ye^{j(\theta+\xi)} \end{bmatrix} \begin{bmatrix} \psi_{11}e^{j\xi} & \psi_{12}e^{j\xi} & 0 \\ -\bar{\psi}_{12} & \bar{\psi}_{11} & 0 \\ 0 & 0 & e^{-j\xi} \end{bmatrix}$$

for any angles  $\theta$  and  $\xi$ , where  $y = \sqrt{1-|\alpha|^2}$  is positive. Another thing is that since the Lie group is not fpf, we cannot use all the 8 degrees of freedom to get a fully-diverse code. Here, we want to sample the Lie group in an appropriate way to obtain fully-diverse subsets. Therefore, to simplify the structure, we set the middle matrix to be  $I_3$  and discuss sets of matrices which are products of 2  $SU(2)$  matrix representations. To get a finite set of  $3 \times 3$  unitary matrices, we choose the entries of  $\Phi$  and  $\Psi$ ,  $\phi_{11}, \phi_{12}, \psi_{11}, \psi_{12}$ ,

as  $P, Q, R, S$ -PSK signals respectively where  $P, Q, R, S$  are positive integers. Define

$$A_{(p,q)}^{(1)} = \begin{bmatrix} e^{j\theta_{p,q}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}}e^{2\pi j \frac{p}{P}} & \frac{1}{\sqrt{2}}e^{2\pi j \frac{q}{Q}}e^{-j\theta_{p,q}} \\ 0 & -\frac{1}{\sqrt{2}}e^{-2\pi j \frac{p}{P}} & \frac{1}{\sqrt{2}}e^{-2\pi j \frac{q}{Q}}e^{-j\theta_{p,q}} \end{bmatrix}$$

$$B_{(r,s)}^{(1)} = \begin{bmatrix} \frac{1}{\sqrt{2}}e^{2\pi j \frac{r}{R}}e^{j\xi_{r,s}} & \frac{1}{\sqrt{2}}e^{2\pi j \frac{s}{S}}e^{j\xi_{r,s}} & 0 \\ -\frac{1}{\sqrt{2}}e^{-2\pi j \frac{r}{R}} & \frac{1}{\sqrt{2}}e^{-2\pi j \frac{s}{S}} & 0 \\ 0 & 0 & e^{-j\xi_{r,s}} \end{bmatrix}$$

The following codes are obtained.

$$\mathcal{C}_{(P,Q,R,S)}^{(1)} = \left\{ A_{(p,q)}^{(1)} B_{(r,s)}^{(1)} \mid \begin{matrix} p \in [0, P), q \in [0, Q), \\ r \in [0, R), s \in [0, S) \end{matrix} \right\} \quad (5)$$

For the code in (5) to be fully-diverse. Both the sets  $A_{(p,q)}^{(1)}$  and  $B_{(r,s)}^{(1)}$  must be fully-diverse. Therefore the angles  $\theta$  and  $\xi$  in the  $A$  and  $B$  matrices must be functions of  $p, q$  and  $r, s$ . Since there are actually no degrees of freedom in them, the number of degrees of freedom in the codes won't be reduced. The simplest function is linear. Therefore, set

$$\theta_{p,q} = 2\pi \left( \frac{p}{P} + \frac{q}{Q} \right) \quad \xi_{r,s} = 2\pi \left( \frac{r}{R} + \frac{s}{S} \right) \quad (6)$$

by which the sets  $\{A_{(p,q,\theta)}^{(1)}\}$  and  $\{B_{(r,s,\xi)}^{(1)}\}$  are fully-diverse when  $\gcd(P, Q) = \gcd(R, S) = 1$ . Therefore, there are all together  $PQRS$  elements in the code (5). Since the channel is used in blocks of 3 transmissions, the rate of the code is  $R = \frac{1}{3} \log_2(PQRS)$ . We can see that the code is a subset of the Lie group  $SU(3)$ . We call it the  $SU(3)$  code.

Note also that the  $e^{-j\theta}$  in the last column of the  $A$  matrix and the  $e^{j\xi}$  in the first row of the  $B$  matrix are used to make the matrices determinant 1. However, in differential unitary space-time code design, we only need the signal matrix to be unitary. Therefore, we can further simplify the structure by abandoning the restriction that both the matrices have unit determinant. Define

$$A_{(p,q)}^{(2)} = \begin{bmatrix} e^{j\theta_{p,q}} & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}}e^{2\pi j \frac{p}{P}} & \frac{1}{\sqrt{2}}e^{2\pi j \frac{q}{Q}} \\ 0 & -\frac{1}{\sqrt{2}}e^{-2\pi j \frac{p}{P}} & \frac{1}{\sqrt{2}}e^{-2\pi j \frac{q}{Q}} \end{bmatrix}$$

$$B_{(r,s)}^{(2)} = \begin{bmatrix} \frac{1}{\sqrt{2}}e^{2\pi j \frac{r}{R}} & \frac{1}{\sqrt{2}}e^{2\pi j \frac{s}{S}} & 0 \\ -\frac{1}{\sqrt{2}}e^{-2\pi j \frac{r}{R}} & \frac{1}{\sqrt{2}}e^{-2\pi j \frac{s}{S}} & 0 \\ 0 & 0 & e^{-j\xi_{r,s}} \end{bmatrix}$$

We obtain the following code with a simpler structure.

$$\mathcal{C}_{(P,Q,R,S)}^{(2)} = \left\{ A_{(p,q)}^{(2)} B_{(r,s)}^{(2)} \mid \begin{matrix} p \in [0, P), q \in [0, Q), \\ r \in [0, R), s \in [0, S) \end{matrix} \right\} \quad (7)$$

With similar argument, set

$$\theta_{p,q} = 2\pi \left( \pm \frac{p}{P} \pm \frac{q}{Q} \right) \quad \xi_{r,s} = 2\pi \left( \pm \frac{r}{R} \pm \frac{s}{S} \right) \quad (8)$$

The code in (7) is not a subset of the Lie group  $SU(3)$  any more since the determinant of the matrices is now  $e^{j(\theta-\xi)}$  which is not 1 in general. However, the matrices in the codes are still unitary matrices. Since any matrix in the code is a

product of two unitary matrices, we call it the AB code. In the following section, we will see that the handy structure of the AB code results in a fast decoding algorithm. The code has the same rate as the code in (5). It is easy to see that any matrix  $U$  in the two codes can be identified by the 4-tuple  $(p, q, r, s)$ .

*Theorem 2 (Calculation of the diversity products):* Let

$$x = e^{2\pi j \left( \frac{p_1 - p_2}{2P} - \frac{q_1 - q_2}{2Q} \right)} \cos 2\pi \left( \frac{p_1 - p_2}{2P} + \frac{q_1 - q_2}{2Q} \right)$$

$$w = e^{2\pi j \left( -\frac{r_1 - r_2}{2R} - \frac{s_1 - s_2}{2S} \right)} \cos 2\pi \left( \frac{r_1 - r_2}{2R} - \frac{s_1 - s_2}{2S} \right).$$

1) For  $U_1, U_2 \in \mathcal{C}^{(1)}$  (defined in (5)-(6)),

$$|\det(U_1 - U_2)| = 2|\text{Im}[1 - \bar{\Theta}x](1 - \bar{\Theta}w)]|$$

$$\text{where } \Theta = e^{-2\pi j \left( \frac{p_1 - p_2}{2P} - \frac{q_1 - q_2}{2Q} + \frac{r_1 - r_2}{2R} + \frac{s_1 - s_2}{2S} \right)}.$$

2) For  $U_1, U_2 \in \mathcal{C}^{(2)}$  (defined in (7)-(8)),

$$|\det(U_1 - U_2)| = 2|\text{Im}[(\Theta_1 - \bar{\Theta}_1 w)(\bar{\Theta}_2 - \Theta_2 x)]|$$

where  $\Theta_1 = e^{2\pi j \left( \pm \frac{p_1 - p_2}{2P} \pm \frac{q_1 - q_2}{2Q} \right)}$ ,  $\Theta_2 = e^{2\pi j \left( \pm \frac{r_1 - r_2}{2R} \pm \frac{s_1 - s_2}{2S} \right)}$ .  $\text{Im}[c]$  indicates the imaginary part of the complex scalar  $c$ .

In the following theorem, the necessary conditions for the codes to be fully-diverse are stated.

From (1) and Theorem 2, the diversity products of the codes are

$$\min |2\text{Im}[1 - \bar{\Theta}x](1 - \bar{\Theta}w)]|^{1/3}/2$$

and

$$\min |2\text{Im}[(\Theta_1 - \bar{\Theta}_1 w)(\bar{\Theta}_2 - \Theta_2 x)]|^{1/3}/2$$

respectively. The minimum is over  $p_1, p_2 \in [0, P), q_1, q_2 \in [0, Q), r_1, r_2 \in [0, R)$  and  $s_1, s_2 \in [0, S)$ . Since the  $x, w, \Theta, \Theta_1, \Theta_2$ , as given in Theorem 2, only depend on the differences  $\delta_p = p_1 - p_2, \delta_q = q_1 - q_2, \delta_r = r_1 - r_2, \delta_s = s_1 - s_2$  instead of  $p_1, p_2, q_1, q_2, r_1, r_2, s_1, s_2$  themselves, the formula inside the minimum, which is the absolute value of the determinant of the difference matrix,  $|\det(U_1 - U_2)|$ , can be written as  $\Delta(\delta_p, \delta_q, \delta_r, \delta_s)$ , which is a function of  $\delta_p, \delta_q, \delta_r, \delta_s$ . The minimum can be calculated over any  $(\delta_p, \delta_q, \delta_r, \delta_s) \neq (0, 0, 0, 0)$  instead. Since  $\delta_p, \delta_q, \delta_r, \delta_s$  can take on  $2P - 1, 2Q - 1, 2R - 1, 2S - 1$  possible values respectively, we only need to calculate the determinants of  $(2P - 1)(2Q - 1)(2R - 1)(2S - 1) - 1 < 16PQRS = 16L$  difference matrices, which is linear in  $L$ , instead of  $L(L-1)/2$  ones in the general case, which is quadratic in  $L$ . Actually, instead of  $16L$ , less than  $8L$  calculations is enough since  $|\Delta(\delta_p, \delta_q, \delta_r, \delta_s)| = |\Delta(-\delta_p, -\delta_q, -\delta_r, -\delta_s)|$ . Therefore, the computational complexity is greatly reduced especially for codes of high rates.

*Theorem 3 (Necessary conditions for fully-diverse):*

1) With the choice of  $\theta, \xi$  in (6), a necessary condition for the code  $\mathcal{C}^{(1)}$  to be fully-diverse is that any two of the integers  $(P, Q, R, S)$  are relatively prime and none of

them are even.

- 2) With the choice of  $\theta, \xi$  in (8), a necessary condition for the code  $\mathcal{C}^{(2)}$  to be fully-diverse is  $\gcd(P, Q) = \gcd(R, S) = 1$  and among the four integers  $(P, Q, R, S)$ , at most one is even, where  $\gcd(P, Q)$  indicates the greatest common divisor of integers  $P$  and  $Q$ .

*Conjecture 1 (Sufficient conditions for fully-diverse):*

The necessary conditions in theorem 3 are also sufficient conditions for the code  $\mathcal{C}^{(1)}$  and  $\mathcal{C}^{(2)}$  to be fully-diverse.

#### IV. DECODING OF THE CODES

Matrices both the codes are products of two basically  $U(2)$  matrices  $A_{(p,q)}^{(i)}$  and  $B_{(r,s)}^{(i)}$  for  $i = 1, 2$  and the  $A$  and  $B$  matrices in  $\mathcal{C}^{(2)}$  have orthogonal design structure with PSK elements. This handy property can be used to get linear-algebraic decoding, which means that the receiver can be made to form a system of linear equations of the unknowns.

The ML decoder for differential USTM given in (3) is equivalent to

$$\begin{aligned} & \arg \min_{p,q,r,s} \|X_\tau - A_{(p,q)}^{(i)} B_{(r,s)}^{(i)} X_{\tau-1}\|_F^2 \\ &= \arg \min_{p,q,r,s} \|A_{(p,q)}^{(i)*} X_\tau - B_{(r,s)}^{(i)} X_{\tau-1}\|_F^2 \end{aligned}$$

Since for code  $\mathcal{C}^{(2)}$ ,  $A^{(2)}$  and  $B^{(2)}$  have orthogonal design structure, the decoder of it can be written equivalently as

$$\arg \min_{p,q,r,s} \|\mathcal{H}\mathcal{X}\|_F^2 \quad (9)$$

where  $\mathcal{H}$  is the following  $3N \times 6$  complex matrix, which only depends on the received signals  $X_\tau$  and  $X_{\tau-1}$ ,

$$\begin{bmatrix} x_{\tau,11} & 0 & 0 & 0 & -\frac{x_{\tau-1,11}}{\sqrt{2}} & -\frac{x_{\tau-1,21}}{\sqrt{2}} \\ 0 & \frac{\bar{x}_{\tau,21}}{\sqrt{2}} & -\frac{\bar{x}_{\tau,31}}{\sqrt{2}} & 0 & -\frac{\bar{x}_{\tau-1,21}}{\sqrt{2}} & \frac{\bar{x}_{\tau-1,11}}{\sqrt{2}} \\ 0 & \frac{\bar{x}_{\tau,31}}{\sqrt{2}} & \frac{\bar{x}_{\tau,21}}{\sqrt{2}} & -x_{\tau-1,31} & 0 & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\ x_{\tau,1N} & 0 & 0 & 0 & -\frac{x_{\tau-1,1N}}{\sqrt{2}} & -\frac{x_{\tau-1,2N}}{\sqrt{2}} \\ 0 & \frac{\bar{x}_{\tau,2N}}{\sqrt{2}} & -\frac{\bar{x}_{\tau,3N}}{\sqrt{2}} & 0 & -\frac{\bar{x}_{\tau-1,2N}}{\sqrt{2}} & \frac{\bar{x}_{\tau-1,1N}}{\sqrt{2}} \\ 0 & \frac{\bar{x}_{\tau,3N}}{\sqrt{2}} & \frac{\bar{x}_{\tau,2N}}{\sqrt{2}} & -x_{\tau-1,3N} & 0 & 0 \end{bmatrix}$$

and  $\mathcal{X} = [e^{-j\theta}, e^{2\pi j \frac{P}{Q}}, e^{-2\pi j \frac{R}{S}}, e^{-j\xi}, e^{2\pi j \frac{P}{Q}}, e^{2\pi j \frac{R}{S}}]^T$  is the vector of unknowns, where  $x_{k,ij}$  is the  $(i, j)$  entry of  $X_k$ . The formula inside the norm in (9) is linear in the PSK unknown signals. Therefore, we can use the sphere decoder for complex channels proposed in [9] with slight modification. The only difference here is that the unknowns  $e^{-j\theta}$  and  $e^{-j\xi}$  are not independent unknown PSK signals but are determined by  $p, q$  and  $r, s$ . Therefore, in the sphere decoding, instead of searching over the intervals for  $e^{-j\theta}$  and  $e^{-j\xi}$ , we calculate

their values by the realizations of  $p, q$  and  $r, s$  respectively based on the choices if  $\theta, \xi$  in (8). Since sphere decoder has an average complexity that is cubic in the code rate and the dimension of the system and at the same time achieving the ML results, we have a fast decoding algorithm for the AB code.

#### V. SIMULATION RESULTS

In this section, the performances of the  $SU(3)$  codes and the AB codes are compared with the group-based codes. The block error rate (bler), which corresponds to errors in decoding the  $3 \times 3$  transmitted matrices, is demonstrated as the error event of interest. The number of receive antennas is 1. Note that the AB code has a fast decoding method while the decoding of the group-based codes and the non-group code needs exhaustive search.

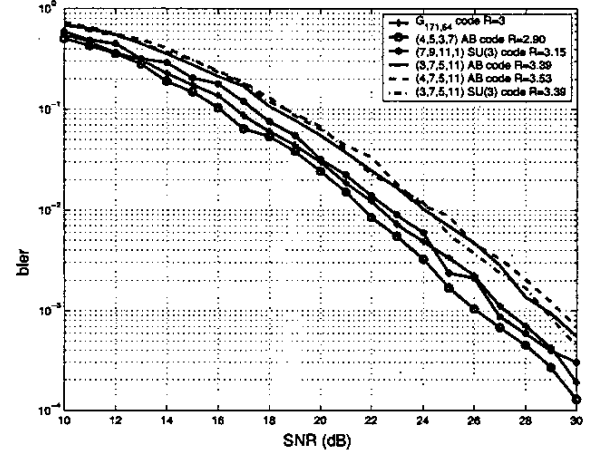


Fig. 1. Comparison of 1 the rate 2.90, (4, 5, 3, 7) AB code with  $\theta = 2\pi(-\frac{P}{Q} + \frac{R}{S})$ ,  $\xi = 2\pi(-\frac{P}{R} - \frac{Q}{S})$ , 2) the rate 3.15, (7, 9, 11, 1)  $SU(3)$  code, 3) the rate 3.39, (3, 7, 5, 11) AB code with  $\theta = 2\pi(\frac{P}{Q} - \frac{R}{S})$ ,  $\xi = 2\pi(-\frac{P}{R} - \frac{Q}{S})$ , 4) the rate 3.53, (4, 7, 5, 11) AB code with  $\theta = 2\pi(-\frac{P}{Q} + \frac{R}{S})$ ,  $\xi = 2\pi(-\frac{P}{R} - \frac{Q}{S})$  and 5) the rate 3.39, (3, 7, 5, 11)  $SU(3)$  code with 6) the rate 3,  $G_{171,64}$  code.

In Fig. 1, we compare two sets of codes. The first set includes the AB code at  $R = 2.90$ , the rate 3  $G_{171,64}$  code and the  $SU(3)$  at  $R = 3.15$ . From the plot we can see that, the AB code is about 1dB better than the  $G_{171,64}$  code. The  $SU(3)$  has about the same performance as the group-based code with a rate 0.15 higher. The second set of codes are the AB codes at rate 3.39 and 3.53 and the  $SU(3)$  code with rate 3.39. The three codes has very close performances. Compared with the performance of the  $G_{171,64}$  code which is shown by the line with circles, the three codes, with rates 0.39, 0.53 and 0.39 higher have performances that are comparable to that of the group-based code. However, the AB codes can be decoded much faster than the  $G_{171,64}$  code and the  $SU(3)$  code.

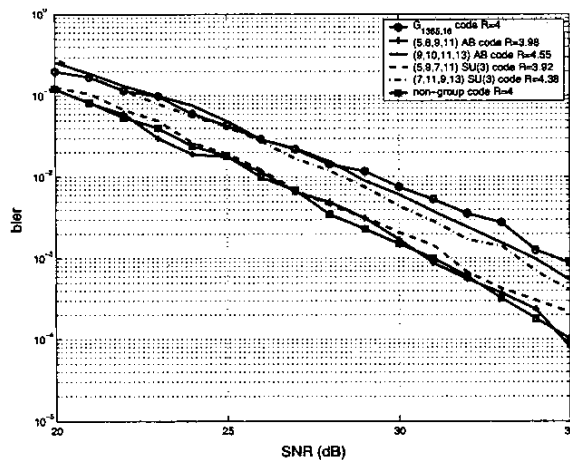


Fig. 2. Comparison of 1) the rate 3.98, (5, 8, 9, 11) AB code with  $\theta = 2\pi(-\frac{p}{p} + \frac{q}{q})$ ,  $\xi = 2\pi(-\frac{r}{r} - \frac{s}{s})$ , 2) the rate 4.55, (9, 10, 11, 13) AB code with  $\theta = 2\pi(-\frac{p}{p} + \frac{q}{q})$ ,  $\xi = 2\pi(-\frac{r}{r} - \frac{s}{s})$ , 3) the rate 3.92, (5, 9, 7, 11),  $SU(3)$  code and 4) the rate 4.38, (7, 11, 9, 13),  $SU(3)$  code with 5) the rate 4  $G_{1365,16}$  code and 6) the rate 4 non-group code.

In Fig 2, we compare the performances of the AB codes at rate 3.98 and 4.55, the  $SU(3)$  code at rate 3.92 and 4.38, with the rate 4 group-based  $G_{1365,16}$  code. From the plot we can see that although with about the same rate, the rate 3.98 AB code and the rate 3.92  $SU(3)$  code perform a lot better than the  $G_{1365,16}$  code. For example at the bler of  $10^{-3}$ , the AB code has an advantage of about 4dB and the  $SU(3)$  code has an advantage of about 3.5dB. With rate 0.38 higher, the  $SU(3)$  code performs 1dB better than the  $G_{1365,16}$  code does. The (9, 10, 11, 13) AB code has a performance that is about the same as the  $G_{1365,16}$  code even with a rate that is 0.55 higher. Also, the plot says that the performance of the AB code and the  $SU(3)$  code are as good as the elaborately designed non-group code given in [1] at rate about 4.

In Fig. 3, we compare the AB code with  $(P, Q, R, S) = (11, 13, 14, 15)$  at rate 4.96 with the rate 5 group-based code. From the plot we can see that the performance of the AB code is much better than the group-based code at about the same rate.

## VI. CONCLUSION

In this paper, we analyze the special unitary Lie group  $SU(3)$  and give two systematic methods to design differential unitary space-time codes that is suitable for systems with 3 transmit antennas and any number of receive antennas. We give formulas from which the diversity products of the two codes can be calculated in a fast way. The AB code can be decoded in polynomial time by the complex sphere decoder while obtaining the ML result. Simulated results show that both codes have better performances than the group-based codes (the only existing methods for  $3 \times 3$  constellations) at

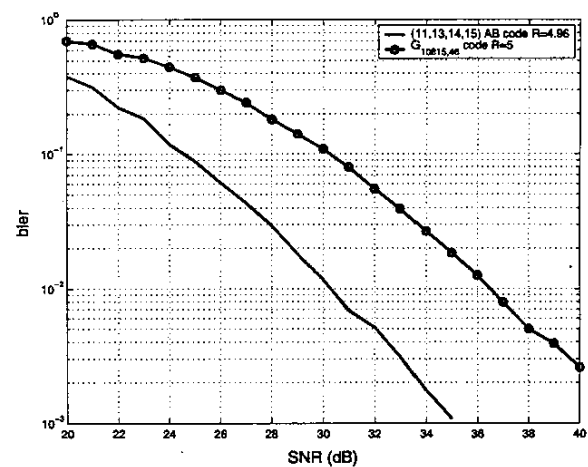


Fig. 3. Comparison of the rate 4.96, (11, 13, 14, 15) AB code with  $\theta = 2\pi(\frac{p}{p} - \frac{q}{q})$ ,  $\xi = 2\pi(-\frac{r}{r} - \frac{s}{s})$  with the rate 5  $G_{10815,46}$  code.

about the same rate. They even have the same performance as the carefully-designed non-group code.

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