On the Capacity of MIMO Broadcast Channel with Partial Side Information

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Abstract— **Since having full channel state information in the transmitter is not reasonable in many applications and lack of channel knowledge does not lead to linear growth of the sum rate capacity as the number transmit antennas increases, it is therefore of interest to investigate transmission schemes that employ only partial CSI. In this paper, we propose a scheme that constructs M random beams and that transmits information to the users with the highest signal-to-noise-plus-interference ratios (SINRs), which can be made available to the transmitter with very little feedback. For fixed M and n increasing, the sum-rate capacity of our scheme scales as $M \log\log n$, which is precisely the same scaling obtained with perfect channel information. We furthermore show that linear increase in capacity can be obtained provided that M does not not grow faster than $O(\log n)$. We also study the fairness of our scheduling scheme and show that, when M is large enough, the system becomes interference-dominated and the probability of transmitting to any user converges to $\frac{1}{n}$, irrespective of its path-loss. In fact, using $M = \alpha \log n$ transmit antennas emerges as a desirable operating point, both in terms of providing linear increase in capacity as well as in guaranteeing fairness..**

I. Introduction

Multiple-antenna communications systems have generated a great deal of interest since they are capable of considerably increasing the capacity of a point to point wireless link. There has also been recent interest in the role of multiple antenna systems in a multi-user network environment, and especially in broadcast and multi-access scenarios. For multiple-input multiple-output (MIMO) broadcast channels the capacity region has been studied in [1], [2], [3] and it has been shown that the sum rate capacity is achieved by *dirty paper coding*.

While the above results suggest that capacity increases linearly in the number of transmit antennas, they all rely on the assumption that the channel is known perfectly at the transmitter. One may speculate whether, as in the point-to-point case, it is possible to get the same gains without having channel knowledge at the transmitter. Unfortunately, it can be proved that, if no channel knowledge is available at the transmitter no matter whether the receivers have full CSI or not, the Gaussian MIMO broadcast channel is degraded and therefore the sum rate capacity does not scale with the number of transmit antennas for high signal to noise ratios (SNRs).

In many applications, however, it is not reasonable to assume that all the channel coefficients to every user can be made available to the transmitter. This is especially true if the number of transmit antennas M and/or the number of users n is large (or if the users are mobile and are moving rapidly). Since perfect channel state information may be impractical, yet no channel state information is useless, it is very important to devise and study transmission schemes that require only partial channel state information at the transmitter. This is the main goal of the current paper.

The scheme we propose is one that constructs M random orthonormal beams and transmits to users with the highest signal-to-noise-plus-interference ratios (SINRs). In this sense it is in the same spirit as the work of [4] where the transmission of one random beam is also proposed. However, our scheme differs in several key respects. First, we send multiple beams (in fact, M of them) whereas [4] sends only a single beam. Second, whereas the main concern in [4] is to improve the proportional fairness of the system (by giving different users more of a chance to be the best user) our scheme aims at capturing as much of the broadcast channel capacity as possible. Fairness is achieved in our system as a convenient by-product.

Based on asymptotic analysis, we show that, for fixed M and n increasing, our proposed scheme achieves a sum-rate capacity of $M \log\log n$. Happily, this is the same as the sum-rate capacity when perfect channel state information is available and so, asymptotically, our scheme does not suffer a loss. One may ask how large may M grow to guarantee a linear increase in capacity? We show that the answer is $M = O(\log n)$.

In schemes (such as ours) that exploit multi-user diversity there is often tension between increasing capacity and fairness. The reason being that the strongest users may dominate the network. Fortunately, we show that in our scheme, provided the number of transmit antennas is large enough, the system becomes interference dominated and so, although close users receive strong signal they also receive strong interference. Therefore it can be shown that, for large enough M, the probability of any user having the highest SINR converges to

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$\frac{1}{n}$. A more careful study of this issue reveals that the choice of $M = \alpha \log n$ transmit antennas is a desirable operating point, both in terms of providing linear increase in capacity as well as in guaranteeing fairness.

This paper is organized as follows. Section 2 introduces our notation and our scheduling algorithm. The asymptotic analysis of the sum rate throughput of our scheme is done in Section 3 when M is fixed and $N = 1$, i.e., single antenna receivers. In Section 4, it is shown that the linear increase in the throughput is retained as long as M is growing not faster than $\log n$. Section 5 deals with the case heterogeneous users and investigates the fairness of our scheduling. Finally Sections 6 and 7 present simulation results and conclude the paper, respectively.

II. Definition and Scheduling Algorithm

In this paper we consider a block-fading multiple antenna channel with M transmit and N receive antennas described by a propagation matrix which is constant during the coherence interval of T. Let $S(t)$ be the $M \times 1$ vector of the transmit symbols at time slot t, and let Y_i be the $N \times 1$ vector of the received signal at the i'th user related by,

$$Y_i(t) = \sqrt{\rho_i} H_i S(t) + W_i, \qquad i = 1, \ldots, n, \quad (1)$$

where ρ_i is the signal to noise ratio of the i'th user and H_i is an $N \times M$ complex channel matrix, known perfectly to the receiver, and W_i is a $N \times 1$ additive noise. The entries of H_i and W_i are i.i.d. complex Gaussian, $CN(0, 1)$, distributed.

In order to exploit having multiple antennas in the transmitter without having full channel knowledge of the channel, the transmitter randomly sends random beams. For simplicity we assume $N = 1$ and we choose M random orthonormal vectors ϕ_m ($M \times 1$) for $m = 1, \ldots, M$ where ϕ_i's are generated according to an isotropic distribution [5]. Further generalization to $N > 1$ receive antennas is omitted in this paper for the sake of brevity [6]. Then at each time instance, the m'th vector is multiplied by the m'th transmit symbol s_m, so that the transmitted signal is,

$$S(t) = \sum_{m=1}^{M} \phi_m s_m, \qquad t = 1, \ldots, T. \quad (2)$$

We also assume that the average transmit power per antenna is one, equivalently, $E\{|s_i|^2\} = 1$, and henceforth the total transmit power is $E\{S^*S\} = M$. After T channel uses, we independently choose another set of orthonormal vectors $\{\phi_m\}$, and so on. From now on, we drop the time index from S and Y, and therefore, the received signal at the i'th receiver is,

$$Y_i = \sum_{m=}^{M} H_i \phi_m s_m + W_i, \qquad i = 1, \ldots, n. \quad (3)$$

We further assume the receiver knows H_i as well as ϕ_i's. Therefore, the i'th receiver ($i = 1, \ldots, n$) can compute the following M signal to interference and noise ratio (SINR) by assuming that s_m is the desired signal and the other s_i's are interference as follows,

$$\text{SINR}_{i,m} = \frac{|H_i \phi_m|^2}{1/\rho_i + \sum_{n \neq m} |H_i \phi_n|^2}, \quad (4)$$

for $m = 1, \ldots, M$. Suppose now each user feeds back its maximum SINR, i.e. $\max_{1 \le m \le M} \text{SINR}_{i,m}$, when the maximum is greater than 1, along with the index m in which the SINR is maximized. Therefore, in the transmitter, instead of randomly assigning the beam to the users, the transmitter assigns s_m to the users with the highest corresponding SINR. The extension of this scheduling to the case with more than one receive antennas is discussed in Section 5.

We also define the sum-rate throughput of the BC channel with this partial side information as the expected value of the total transmission rates to all users and denoted by R. Furthermore we call a scheduling fair if the probability of choosing users with different signal to noise ratios (ρ_i) is equal. Clearly in an interference dominant system, this definition is equivalent to giving the same rate to all users irrespective to their signal to noise ratios.

III. Sum Rate Throughput: $N = 1$, M is fixed

In this section we obtain lower and upper bounds for the sum rate capacity when M is fixed, $N = 1$ and n is going to infinity. Using M random beams and sending to the users with the highest SINRs, we can bound the sum rate throughput R, as

$$R \le E\left\{ \sum_{m=1}^{M} \log\left(1 + \max_{i=1,\ldots,n} \text{SINR}_{i,m}\right) \right\}, \quad (5)$$

where this is an upper bound since we ignored the probability that user i be the maximum SINR user twice (if this is the case, the transmitter has to choose another user with SINR less than the maximum SINR which decreases the capacity).

On the other hand, in [6], the following lower bound for the sum rate throughput is proved

$$R \ge M\left(1 - \{\Pr\{\text{SINR}_{i,1} \le 1\}\}^n\right) \times$$
$$E\left\{ \log\left(1 + \max_{i=1,\ldots,n} \text{SINR}_{i,1}\right) \Big| \max_{i=1,\ldots,n} \text{SINR}_{i,1} \ge 1 \right\}. \quad (6)$$

As we shall show later, the lower and upper bounds for the sum rate capacity become tight for sufficiently large n and when $\lim_{n\to\infty} \frac{M}{\log n} = 0$. In this case, conditioning on $\max_{1 \le i \le M} \text{SINR}_{i,m} \ge 1$ in Eq. (6) can be replaced by $\max_{1 \le i \le M} \text{SINR}_{i,m} \ge \eta$ where η is a constant independent of n and the bounds remain tight. This implies that the receiver is only required to feedback its maximum SINR if it is greater than η along with the index m corresponding to the signal. Therefore the amount of feedback here will be $2n\Pr\{\max_{1 \le m \le M} \text{SINR}_{i,m} \ge \eta\}$ real numbers and M integers (at most). However, in the case with full CSI in the transmitter, the amount of feedback is $2nM$ real numbers which is at least M times bigger than what we need in our scheme. Furthermore the complexity of our scheme is much less than the proposed schemes to implement Dirty paper coding with full CSI using nested lattices or trellis precoding [7], [8].

In order to evaluate the lower and upper bounds, we have to obtain the distribution of $\mathrm{SINR}_{i,m}$. As mentioned earlier, $|H_i\phi_m|^2$'s are i.i.d. over m (and also over i) with $\chi^2(2)$ distribution. Thus, the probability distribution function (PDF) of $\mathrm{SINR}_{i,m}$, $f(x)$, can be derived as [6],

$$f(x) = \frac{e^{-x/\rho}}{(1+x)^M}\left(\frac{1}{\rho}(1+x) + M - 1\right). \quad (7)$$

We can also calculate the cumulative distribution function (CDF) of $\mathrm{SINR}_{i,m}$, $F(x)$, as,

$$F(x) = 1 - \frac{e^{-x/\rho}}{(1+x)^{M-1}}, \qquad x \geq 0. \quad (8)$$

Since $\mathrm{SINR}_{i,m}$ for $i = 1, \ldots, n$, are i.i.d. random variables, the CDF of $\max_{1\leq i\leq n} \mathrm{SINR}_{i,m}$ for $m = 1 \ldots, M$ is $F^n(x)$. Using the obtained CDF we can now evaluate bounds for the sum rate capacity of our proposed scheme [6]:

Lemma 1: For any ρ, M, and n, the sum rate capacity of the randomly chosen beamforming, R, satisfies:

$$M\int_1^\infty \log(1+x)nf(x)F^{n-1}(x)dx \leq R$$
$$\leq M\int_0^\infty \log(1+x)nf(x)F^{n-1}(x)dx. \quad (9)$$

where $f(x)$ and $F(x)$ are as defined in (7) and (8), respectively.

Lemma 1 can be used to evaluate the sum rate capacity for any n, ρ, and M. However in many systems ρ and M are fixed, but n (the number of users) is large. In what follows, we will focus on the scaling laws of the rate sum rate for large n.

In fact the asymptotic behavior of the distribution of the maximum of n i.i.d. random variables has been extensively studied in literature [9], [10]. In [6], it is shown that

$$\Pr\left\{\left|\frac{\max_{1\leq i\leq n}\mathrm{SINR}_{i,m}}{\rho\log n} - 1\right| \leq O\left(\frac{\log\log n}{\log n}\right)\right\}$$
$$= 1 - O\left(\frac{1}{\log n}\right). \quad (10)$$

Remark 1: Eq. (10) shows that when M is fixed and n increases, the maximum SINR behaves like $\rho\log n + O(\log\log n)$. On the other hand, from the expression for the SINR defined in (4), it is clear that the numerator is a $\chi^2(2)$ random variable and the interference terms constitute a $\chi^2(2M-2)$ random variable. It is well known that the maximum of n i.i.d. $\chi^2(2)$ behaves like $\log n$ for large n [6]. One may then suspect that $\max_{i=1,\ldots,n} \mathrm{SINR}_{i,m}$ should behave like $\frac{\log n}{\frac{1}{\rho}+M-1}$, arguing that when the numerator takes on its maximum the denominator takes on its average value. What is interesting about Eq. (10) is that this heuristic argument is not true. It turns out that $\max_{i=1,\ldots,n} \mathrm{SINR}_{i,m}$ is achieved when the numerator behaves as $\log n$ *and* the interference terms are arbitrarily small, this yielding the behavior $\rho\log n$. We can now state the following Theorem to prove the asymptotic linear growth in the sum rate throughput when M is fixed [6].

Theorem 1: Let M and ρ be fixed and $N = 1$. Then

$$\lim_{n\to\infty}\frac{R}{M\log\log n} = 1 \quad (11)$$

Remark 2: It is not hard to obtain the next order term in R as follows

$$R \geq M\log(\rho\log n - \rho M\log\log n) + o(\log\log\log n). \quad (12)$$

Theorem 1 states that for fixed M as n grows to infinity the sum rate throughput scales like $M\log\log n$. Interestingly [6], we proved that $M\log\log n$ is in fact the best sum rate capacity that can be achieved with full knowledge of the channel using Dirty Paper coding [2], [1], [3]. Therefore, asymptotically we are not losing anything in terms of the sum rate throughput for the case where M is fixed. This in fact raises the question of how far can we increase M and still have linear increase in the sum rate that will be answered in the next section.

IV. How Far Can M Go to Retain Linear Increase in R?

In this section, we consider the case where the number of transmit antennas M is allowed to grow to infinity. Similar to the previous section, we assume each receiver has a single antenna and the total average transmit power is M, i.e. the average transmit power per antennas is one.

Theorem 2: If $M = \frac{\log n}{\log(1+c)} + O(\log\log n)$, where c is a positive constant. Then,

$$\Pr\left\{c - O\left(\frac{\log\log n}{\log n}\right) \leq \max_{1\leq i\leq n}\mathrm{SINR}_{i,m} \leq c\right\}$$
$$\geq 1 - O\left(\frac{1}{\log^3 n}\right). \quad (13)$$

Consequently,

$$\lim_{n\to\infty}\frac{R}{M\log(1+c)} = 1 \quad (14)$$

Theorem 2 shows that increasing M improves the slope of the sum rate linearly but reduces the argument of the logarithm. We now show that increasing M at a rate stronger than $O(\log n)$ results in sub-linear growth of the sum rate. In [6], it is further proved that if $\lim_{n\to\infty}\frac{M}{\log n} = \infty$, then $\lim_{n\to\infty}\frac{R}{M} = 0$. Therefore we retain a linear growth in the sum rate for a number of transmit antennas growing as $O(\log n)$.

V. Fairness in the Scheduling

So far, we have assumed a homogeneous network in the sense that the SNR for all users was equal, namely $\rho = \rho_i$, $i = 1, \ldots, n$. In practice, however, due to the different distances of the users from the base station and the corresponding different path losses, the users will experience different SNRs so that ρ_i's will not be identical. Such networks are called heterogeneous.

In heterogeneous networks, there is usually tension between the gains obtained from employing multi-user diversity and the fairness of the system. A fortunate consequence of our random multi-beam method is that, if the number of transmit antennas is large enough then the system becomes interference dominated. In this case, being the best user will depend not so much on how close one is to the base station, but rather on how ones channel vector H_i aligns with the closest beam direction ϕ_m, $m = 1, \ldots, M$. Therefore, one would expect that the probability that any user is the strongest will not depend on its SNR ρ_i.

In what follows we will make this observation more precise. We will show that if the number of transmit antennas M grows faster than or equal to $O(\log n)$ then the system will be fair, thus we achieve maximum throughput and fairness simultaneously.

Denoting the signal to noise ratio of the i'th user by ρ_i, then the PDF of $\text{SINR}_{i,m}$ is as in (7) where ρ is replaced with ρ_i. We are interested in computing the probability of transmitting the m-th signal to the $i'th$ user, i.e.

$$\begin{aligned} P_i &= P_i^m \\ &= \int_0^\infty \int_0^{x_i} \cdots \int_0^{x_i} f_i(x_i) \prod_{j=1, j\neq i}^{n} f_j(x_j) dx_1, \ldots, dx_n. \end{aligned}$$

Note that due to the fact that $\text{SINR}_{i,m}$ for $m = 1, \ldots, M$ has identical distribution, P_i^m does not depend on the index m and $P_i = P_i^m$ for $m = 1, \ldots, M$. In [6], a lower bound for the probability of choosing the strongest user with the highest SNR is derived as,

$$P_{\rho_{\max}} \leq \frac{e^{(\frac{1}{\rho_{\min}} - \frac{1}{\rho_{\max}})(e^{\frac{2\log n}{M-1}} - 1)}}{n} + \frac{1}{n^2} \tag{15}$$

where $\rho_{\min}$ and $\rho_{\max}$ denote to the lowest and highest SNR. Therefore, we can state that if $\frac{M}{\log n} = \alpha$ then by increasing the signal to ratio of the users $P_{\rho_{\max}} \to \frac{1}{n}$, and so the system becomes more and more fair. Alternatively, if we fix the SNR and increase α, $P_{\rho_{\max}} \to \frac{1}{n}$ and the systems becomes fair.

VI. Simulation Results

In this section we verify our asymptotic results with simulations and numerical evaluation. As Lemma 1 states, bounds on the sum rate throughput can be evaluated for any n, M, and ρ. We also proved that in Theorem 1 and 2 the upper bound is tight when $M \leq \alpha \log n$ which is the region that we are interested in, therefore, we plot the upper bound in Lemma 1 as a good approximation for the sum rate capacity. Fig. 1 and 2 show the capacity versus the number of transmit antennas M, for different SNR's. Clearly for $M \leq 4$ the curve behaves linearly and as M becomes $\log n \approx 4$ the throughput curves become saturated.

Using simulations, we also compare the fairness of our scheduling with multiple transmit antennas with that of the case with one antenna in the base station $M = 1$, in which the best scheduling strategy (in terms of maximizing the sum rate capacity) is to transmit to the user with the maximum SNR.

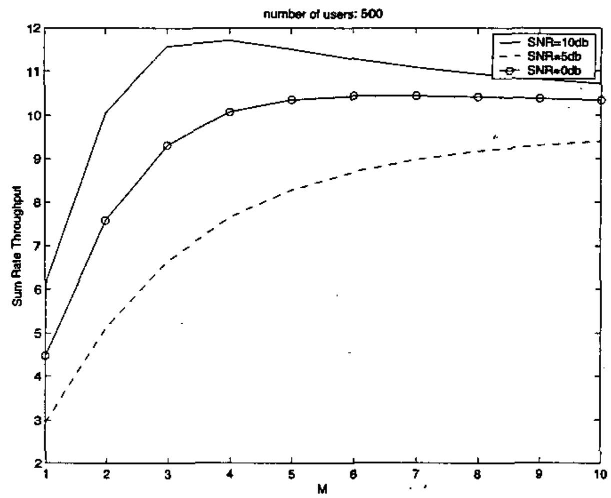

Fig. 1. Sum rate capacity versus the number of transmit antennas for different SNRs and $n = 500$.

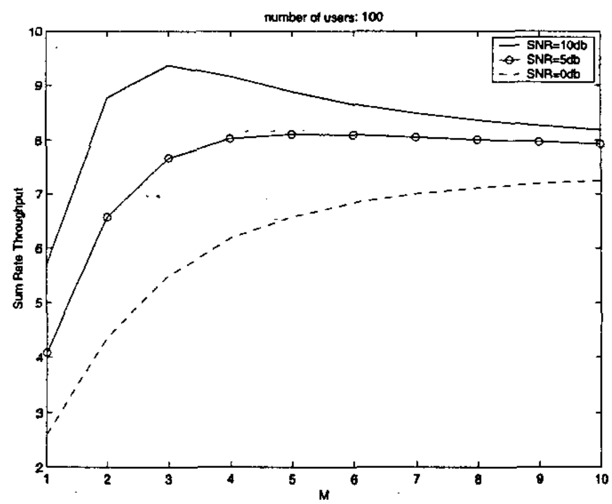

Fig. 2. Sum rate capacity versus the number of transmit antennas for different SNRs and $n = 100$.

Suppose users have SNRs uniformly distributed from 6 dB to 15 dB, therefore the users corresponding to the SNR of 15dB and 6dB are the strongest and the weakest users, respectively. Fig. 3 shows the number of times that each user with the corresponding SNR is chosen out of 50000 iterations. On the other hand, Fig. 4 shows the fairness of our proposed algorithm by using $M = 5(\approx \log n)$ antenna in the base station. As Fig. 3 and 4 show, the fairness has been significantly improved by using multiple transmit antennas. For instance, the ratio of the number of times that the strongest user is chosen to the number of times that the weakest user is chosen, is 700 for the case with $M = 1$ as opposed to 2.5 for the case with $M = 5$

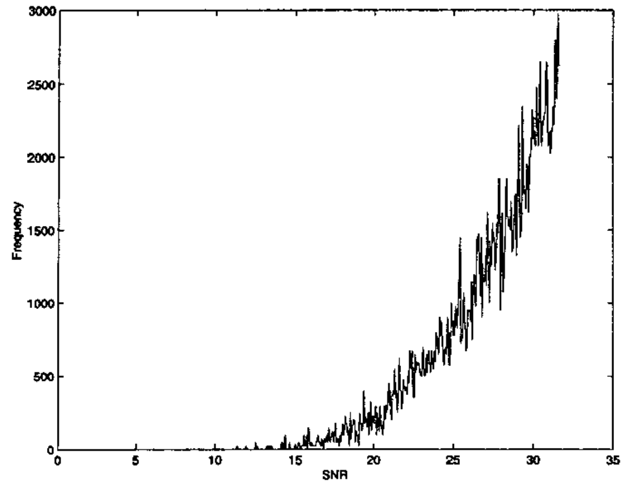

Fig. 3. The number of times that each user with the corresponding SNR is chosen for 50000 iterations with $M = 1$ and $n = 500$.

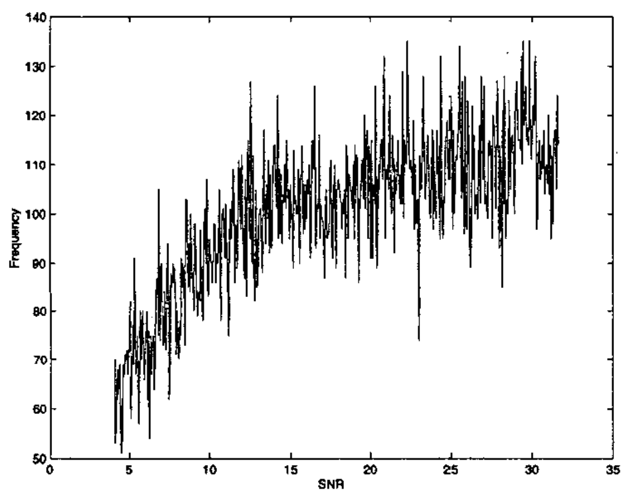

Fig. 4. The number of times that each user with the corresponding SNR is chosen for 10000 iterations with $M = 5$ and $n = 100$.

using our scheduling.

VII. Conclusion

We proposed an scheduling for a MIMO broadcast channel which requires a little feed back from the receivers. We showed that the sum rate throughput of this scheme is $M \log\log n$ where M is fixed and n is sufficiently large. It is further shown that $M \log\log n$ is the best that one can do with the full knowledge of the channel in the receiver. We concluded that by using $M = \alpha \log n$ we can guarantee the fairness and linear increase in the sum rate throughput. It is worth noting that the results here can be extended to the case with more than one receive antennas.

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