On the Asymptotic Capacity of MIMO Broadcast Channels with Partial CSI

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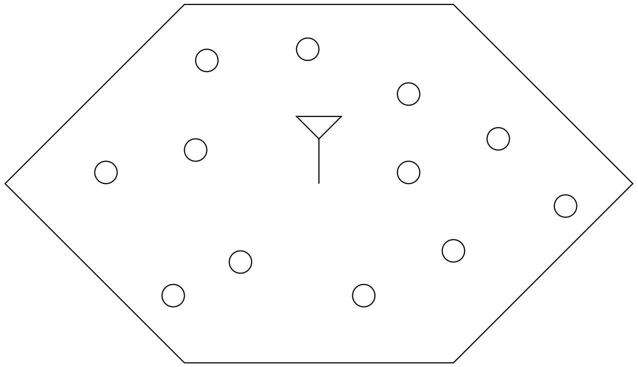
IEEE Communication Theory Workshop

Mesa, Arizona, April 8, 2003

Outline

- **Introduction**
- **MIMO broadcast channels**
  - asymptotic sum-rate capacity with and without CSI
  - propose a scheme that requires very little CSI
  - asymptotic analysis
  - fairness issues
- **Conclusion**

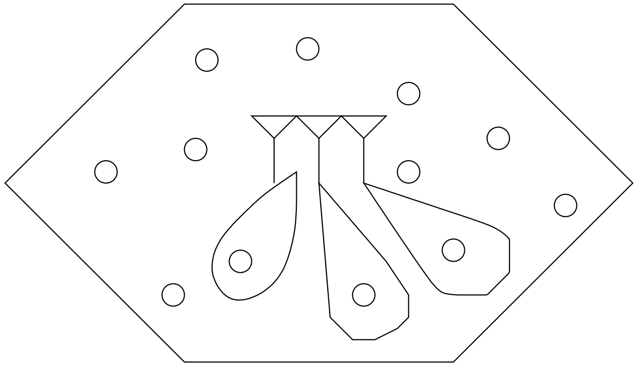
Cellular Networks



- capacity per cell is fixed (depends on bandwidth and transmit power)
- TDMA, FDMA, CDMA all essentially give same capacity
- to increase network capacity cell sizes are shrinking

Under ideal conditions, there exist sophisticated techniques where capacity scales by $\log n$ (n is number of users)

Increasing Cell Capacity with Multiple Antennas



- idea is to exploit *spatial diversity* to transmit simultaneous beams to more than one user
- requires knowledge of spatial location of different users at base station (also line-of-sight environment)
- some companies attempted this in the 90's (e.g., ArrayComm) but it never really caught on

Multiple Antennas in Single User Links

Consider a single user link with M transmit and N receive antennas. Assume that the environment is rich-scattering.

- Transmitter and receiver know the channel:

$$C = \min(M, N) \log \mathrm{SNR} + O(1)$$

- Only receiver knows the channel (Foschini '97, Telatar '97):

$$C = \min(M, N) \log \mathrm{SNR} + O(1)$$

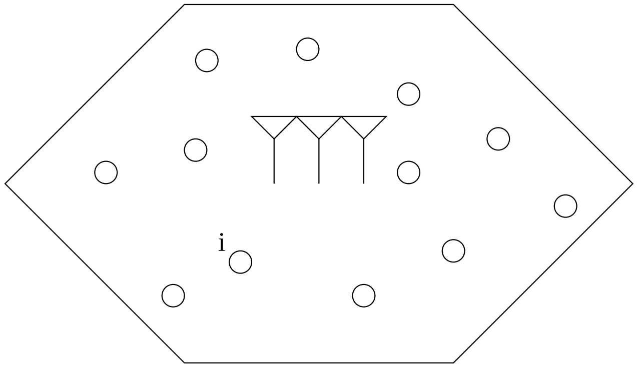
- Neither receive nor transmitter knows the channel (Zheng and Tse '01, Hassibi and Marzetta '01):

$$C = \min(M, N) \left(1 - \frac{\min(M, N)}{T}\right) \log \mathrm{SNR} + O(1)$$

where T is the coherence interval of the channel

Multiple Antennas in Broadcast Channels

Suppose now we have M transmit antennas at the base station.



The channel to the i-th user is given by an M-dimensional vector:

$$h_i = \left[\begin{array}{cccc} h_{i1} & h_{i2} & \ldots & h_{iM} \end{array} \right]$$

The h_i are independent random vectors, whose distribution depends on the fading environment.

For Rayleigh fading the entries of h_i are iid complex Gaussian.

Question: *What is the capacity of such a multi-antenna broadcast channel?*

Answer depends on what the transmitter knows about the channels.

Channel Coefficients Known to Transmitter (Shamai and Caire '02)

$$C = \max_{\Pi, \mathrm{tr} \sum_i R_i = P} \sum_{i=1}^{n} E \log\det\left(I_M + \left(\sigma^2 I_M + h_i^* \sum_{j=1}^{i-1} R_j h_i \right)^{-1} h_i^* R_i h_i \right)$$

- Achieved by what is known as *dirty paper coding* (Costa '80).
- Asymptotically, for fixed M and $n \to \infty$, this behaves as

$$C = M \log n + O(1)$$

- The problem is that this requires full channel knowledge at the transmitter and is very sensitive to channel errors
- Dirty paper coding is very computationally-intensive (both the transmitter and receiver need to do vector quantization), although there exist simplified schemes such as Tomlinson-Harashima precoding with reasonable performance

Channel Coefficients Unknown Transmitter

$$C = \log M + O(1)$$

independent of n! (Assuming fixed power per transmit antenna.)

Thus, lack of knowledge of the channel coefficients brings us down from $M \log n$ to $\log M$.

- full channel-state-information (CSI) unreasonable
- multiple antennas do not buy us much if no CSI available

What to do?

Send Random Beams

Choose M random orthonormal vectors ϕ_m, $m = 1, \ldots, M$ (according to an isotropic distribution). At time t, the m-th vector is multiplied by the signal $s_m(t)$ ao that the transmitted signal is

$$S(t) = \sum_{m=1}^{M} \phi_m s_m(t), \qquad t = 1, \ldots, T$$

where T is less than the coherence interval of the channel.

After T channel uses we independently choose another isotropic set of orthonormal vectors $\{\phi_m\}$, and so on.

In other words, we are transmitting M random beams.

Each receiver $i = 1, \ldots, n$ therefore can compute the following M SINRs

$$\mathrm{SINR}_{im} = \frac{|h_i \phi_m|^2}{1/\mathrm{SNR} + \sum_{n \neq m} |h_i \phi_n|^2}$$

Of course, on average the SINR is (roughly) $\frac{1}{1/\mathrm{SNR}+M-1}$, and so if we randomly assign beams to users we get

$$C \approx M \log\left(1 + \frac{1}{1/\mathrm{SNR} + M - 1}\right) < \frac{M}{1/\mathrm{SNR} + M - 1},$$

which is pretty lousy.

So what is the point?

Exploit Multi-User Diversity

Suppose now each user (or, in fact, only those who get favorable SINRs) feeds back to the transmitter its best SINR.

Rather than randomly assign the beams, the transmitter assigns signal s_m to the user with the best SINR for that signal. Therefore

$$C = E \sum_{m=1}^{M} \log \left(1 + \max_{i=1,\ldots,n} \frac{|h_i \phi_m|^2}{1/\mathrm{SNR} + \sum_{n \neq m} |h_i \phi_n|^2} \right)$$

Due to the symmetry of all the random variables involved:

$$C = ME \log \left(1 + \max_{i=1,\ldots,n} \frac{|h_i \phi_1|^2}{1/\mathrm{SNR} + \sum_{n \neq 1} |h_i \phi_n|^2} \right)$$

Note that the random variables

$$\frac{|h_i \phi_1|^2}{1/\mathrm{SNR} + \sum_{n \neq 1} |h_i \phi_n|^2}$$

are ratios of independent Gaussians, and themselves are independent.

Maximum of n IID Random Variables

Theorem 1 (Frechet, 1927, v. Mises 1947, Uzgoren 1954) *Let x_i, $i = 1, \ldots, n$ be iid random variables with distribution $p(x)$. Then if,*

$$\lim_{x \to \infty} -\frac{p(x)}{p'(x)} = c > 0,$$

then

$$Prob\left(\frac{|x_{\max} - \frac{1}{c}\log n|}{\log\log n} > \epsilon \right) < \frac{\delta}{(\log n)^{\alpha}}.$$

In our case, the condition of the theorem is met with $c = 1/\mathrm{SNR}$.

Therefore, for M fixed and $n \to \infty$

$$C = M \log\log n + O(\log\log\log n)$$

Thus, multi-user diversity buys us a lot!

Some Remarks

- Using partial CSI, our scheme achieves $M \log\log n$, as opposed to $M \log n$ for full CSI and $\log M$ for no CSI.
- If we have log-normal shadowing

$$C = M\sqrt{\log n} + O(\sqrt{\log\log n}).$$

- Compared to the full CSI case where each user had to feedback M channel coefficients, here each user need only feed back its *best* SINR and the corresponding index.
  - In fact, not all users need to do this. If only users whose best SINR exceeds a threshold, η, say, feed it back then

$$C \geq \left(1 - F^{Mn}(\eta)\right) M \log\log n + O(\log\log\log n),$$

    where $F(\cdot)$ is the CDF of the SINR. This reduces the feedback by a factor of $\frac{1}{1-F(\eta)}$.

How Large can M Be?

Our result required that M be fixed and $n \to \infty$.

In this case, $\mathrm{SINR}_{\max}$ was obtained when $\sum_{j \neq m} |h_j \phi_m|^2 \approx 0$ and $|h_i \phi_m|^2 \approx \log n$.

However, in practice, we will have a large (but finite n) and so it is useful to know how large M can be to retain a linear increase in capacity.

Result: Let $M = \alpha \log n$. Then as $n \to \infty$, we have $\mathrm{SINR}_{\max} \to 1/\alpha$ and

$$C = M \log \left(1 + \frac{1}{\alpha}\right) + O(1)$$

Moreover, if $\frac{M}{\log n} \to \infty$, then $\frac{C}{M} \to 0$.

What if We Add Multiple Receive Antennas?

Suppose we have $N > 1$ receive antennas. There are three possibilities:

- assign one beam to user (feed back capacity)

$$C = M \log \log n + O(\log \log \log n)$$

- assign M/N beams to user (feed back capacity)

$$C = M \log \log n/N + O(\log \log \log n)$$

- assign one beam to user (treat each antenna independently)

$$C = M \log \log nN + O(\log \log \log n)$$

The last is the best. In any case, multiple receive antennas do not substantially increase the network capacity.

- However, if some fraction of the users have N receive antennas they will receive N-times the rate.

Fairness of the System

- So far we have assumed that the network is homogenous
- Most networks are heterogenous, in the sense that the SNRs for the different users are different
- Thus, if we transmit to the most favorable users, the system may be dominated by the users with the highest SNR
- This is certainly true for single-transmit-antenna systems

Single Antenna Multi-User Fairness

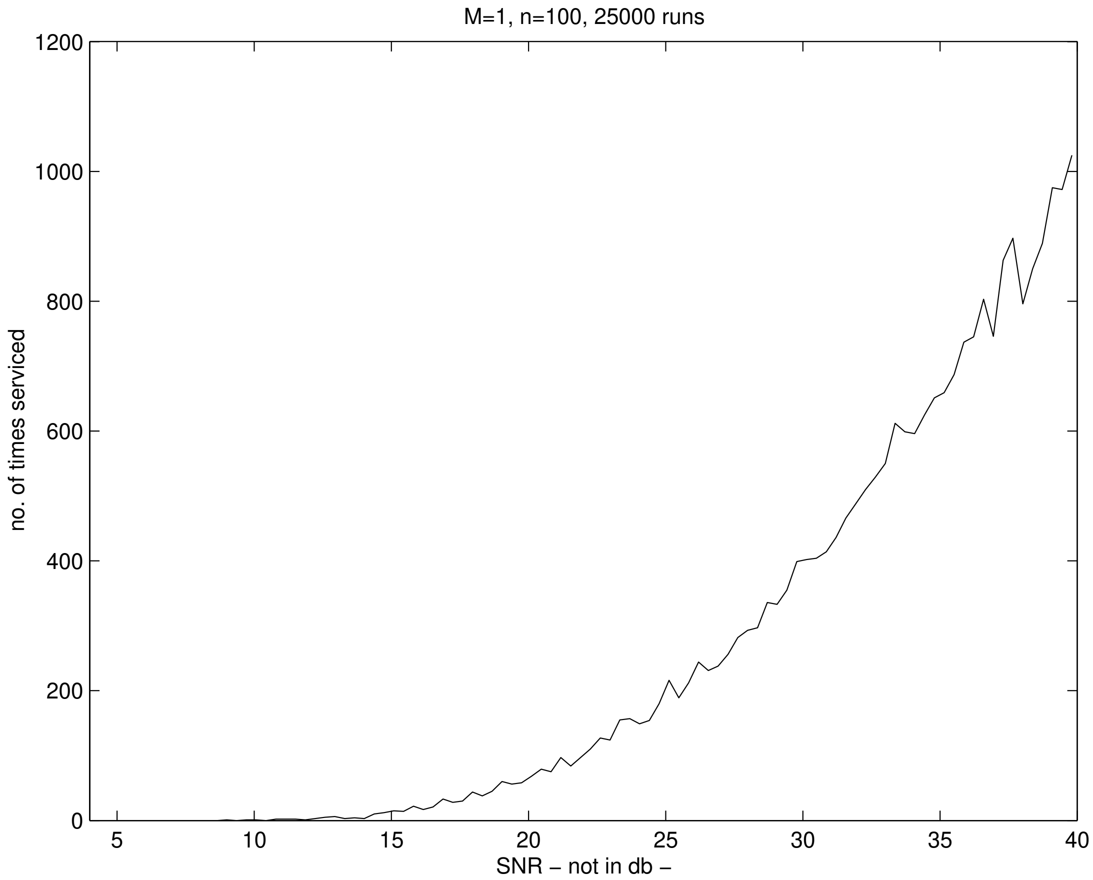


Figure 1: $M = 1$, $n = 100$, SNR 4-40 (6-16db), 25000 runs

Fairness - Continued

In a heterogenous network, the SINRs now become

$$\mathrm{SINR}_{im} = \frac{|h_i \phi_m|^2}{1/\mathrm{SNR}_i + \sum_{n \neq m} |h_i \phi_n|^2}$$

Note that, if the SNR_is are high enough, or if the number of transmit antennas is large enough, then the system is interference-dominated which implies *built-in fairness.*

Result:

$$P(\text{choosing user with } \mathrm{SNR}_{min}) \geq \frac{1}{n} e^{-\left(\frac{1}{SNR_{min}} - \frac{1}{SNR_{max}}\right)\frac{\log n}{M}},$$

This further illustrates the benefits of having $M = \alpha \log n$.

Multi-Antenna Multi-User Fairness

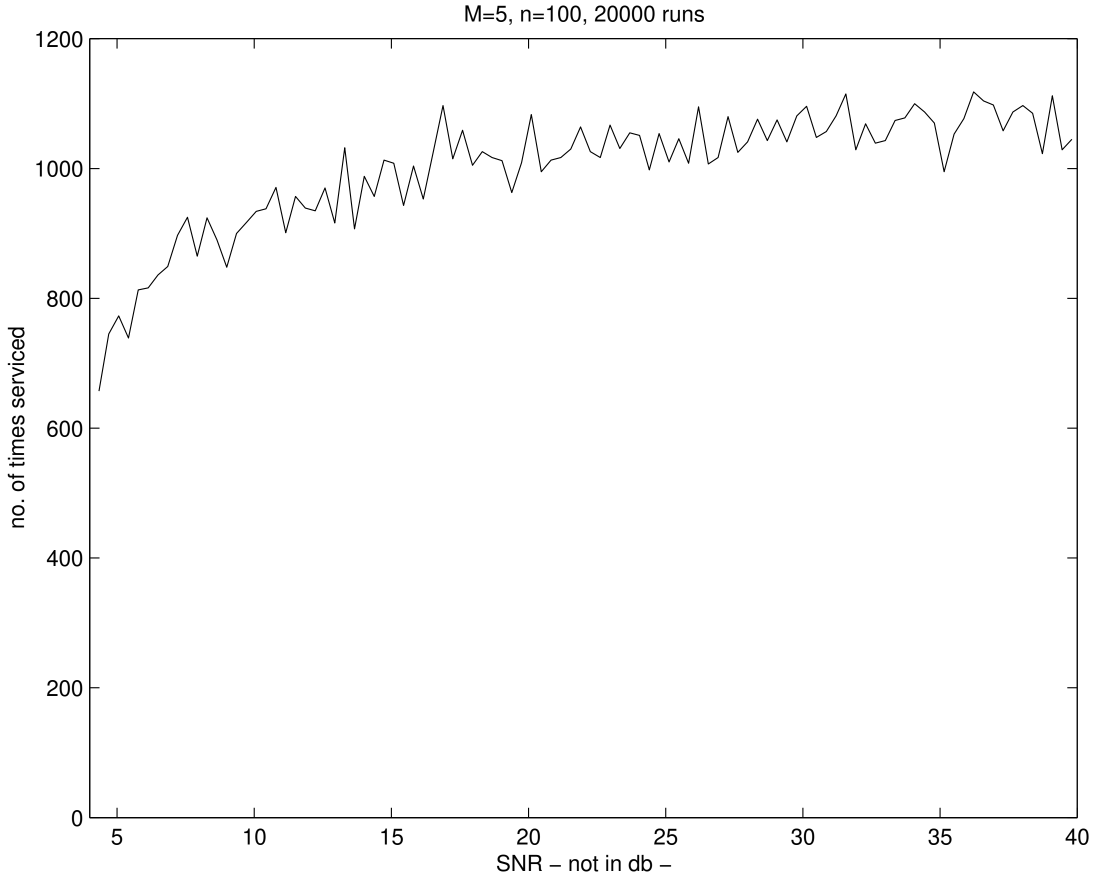


Figure 2: $M = 5$, $n = 100$, SNR 4-40 (6-16db), 20,000 runs

Summary and Final Remarks

- proposed a simple scheme requiring very little feedback that allows linear increase in capacity in number of transmit antennas
- for M fixed and $n \to \infty$, capacity scales as $M \log\log n$
- capacity increases linearly provided number of transmit antennas does not exceed $O(\log n)$
- capacity increases in the presence of shadowing ($M\sqrt{\log n}$)
- the scheme exploits multi-user diversity, but also has built-in fairness provided the number of transmit antennas is $O(\log n)$