Homework Set #2

1. *(Sampling theorem for random processes.)* Recall from EE111 that if $x(\cdot)$ is a deterministic bandlimited signal with bandwidth f_m, then it can be recovered from its samples according to the formula

$$x(t) = \frac{1}{T}\sum_{k=-\infty}^{\infty} x(kT)\operatorname{sinc}\frac{t-kT}{T},$$

where $\operatorname{sinc}(\alpha) = \frac{\sin \pi\alpha}{\pi\alpha}$ and the sampling period T is such that $T < \frac{1}{2f_m}$.

In this problem we would like to study whether a similar sampling theorem holds for random processes. To this end, we call the random process $x(\cdot)$ *bandlimited* if its power spectral density $S_x(f)$ is bandlimited to a bandwidth f_m, i.e.,

$$S_x(f) = 0, \quad \text{for all } |f| > f_m.$$

   (a) Show that the autocorrelation function $R_x(\tau)$ satisfies the sampling theorem and give the precise statement.

   (b) Use part (a) to show that, for all $T < \frac{1}{2f_m}$,

$$E\left(x(t) - \frac{1}{T}\sum_{k=-\infty}^{\infty} x(kT)\operatorname{sinc}\frac{t-kT}{T}\right)^2 = 0.$$

   Interpret the above result.

2. Consider the setting of Fig. 1. $u(\cdot)$ and $v(\cdot)$ are zero-mean wide sense stationary processes with autocorrelation functions $R_u(\tau) = \delta(\tau)$ and $R_v(\tau) = \frac{N_0}{2}\delta(\tau)$. The goal is to estimate the process $s(\cdot)$, obtained from passing $u(\cdot)$ through a known stable LTI system with transfer function $L(f)$. The only process available to us is $y(\cdot)$, the output of the known stable LTI system $H(f)$ which is further corrupted by the additive noise $v(\cdot)$. We would like to design a stable LTI filter, $K(f)$, which constructs a "good" estimate $\hat{s}(\cdot)$ of $s(\cdot)$ using the observable process $y(\cdot)$.

   (a) Find the power spectral density of the estimation error process $\tilde{s}(\cdot) = s(\cdot) - \hat{s}(\cdot)$.

   (b) Determine the filter $K(f)$ such that it minimizes $E\tilde{s}(t)^2$, the power of the estimation error process.

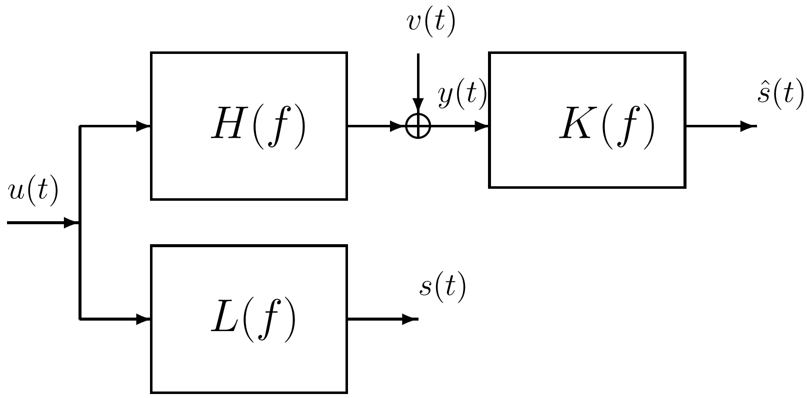

Figure 1: *A linear estimation problem.*

(c) Show that the resulting minimum estimation error power is given by

$$E\tilde{s}(t)^2 = \int_{-\infty}^{\infty} \frac{|L(f)|^2}{1 + \frac{2}{N_0}|H(f)|^2} df.$$

(d) Assume that $H(f) = 1$ and show that

$$\frac{E\tilde{s}(t)^2}{Es(t)^2} = \frac{N_0/2}{N_0/2 + 1}.$$

3. Problem 1.20 in Haykin.

4. Problem 1.26 in Haykin.